MATH70062: Lie Algebras

Lecturer: Ambrus Pal

Scribe: Sidharth Hariharan

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Contents

Chapter 1

An Introduction to the Theory of Lie Algebras

While many of the definitions and constructions we shall see in this course can easily be adapted to any field, we will work over $\mathbb{C}$ for simplicity, unless otherwise stated.

1.1 Important Definitions and First Examples

We will begin by defining the fundamental objects of study in this course. We will then provide some examples of these objects and discuss means of constructing them.

1.1.1 Algebras

We begin by recalling the notion of a bilinear map.

Definition 1.1.1 (Bilinear Map). Let V and W be vector spaces. We say that a map $f : V \times W \to \mathbb{C}$ is **bilinear** if it is linear in each argument. That is, for all $v, v' \in V$, $w, w' \in W$ and $\lambda \in \mathbb{C}$, we have

$$f(v + v', w) = f(v, w) + f(v', w)$$
$$f(v, w + w') = f(v, w) + f(v, w')$$

$$f(\lambda v, w) = \lambda f(v, w) = f(v, \lambda w)$$

We will be particularly interested in bilinear maps from a vector space to itself.

Definition 1.1.2 (Algebra). An **algebra** is a vector space A equipped with a bilinear map $\cdot : A \times A \to A$.

Convention. Given any algebra A, we will often refer to the corresponding bilinear map $\cdot$ as the **multiplication** map of A, and denote $\cdot(x, y)$ as simply $x \cdot y$ or even xy (where the definition of $\cdot$ is clear from the context) for any $x, y \in A$.

There are many different kinds of algebras. We will be particularly interested in Lie algebras and associative algebras.

Definition 1.1.3 (Associative Algebras). We say that an algebra A is **associative** if the multiplication map $\cdot$ is associative. That is, for all $x, y, z \in A$, we have

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

We have all seen associative algebras before.

Example 1.1.4 (The Matrix Algebra). The set $\mathrm{M}_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$ forms an associative algebra under matrix multiplication, known as the Matrix Algebra.

We will come back to associative algebras soon enough. We will now define the main object of study in this module.

Definition 1.1.5 (Lie Algebras). A **Lie algebra** is an algebra L whose bilinear map $[\cdot, \cdot] : L \times L \to L$ satisfies the following properties:

1. For all $x \in L$, we have $[x, x] = 0$.
2. For all $x, y, z \in L$, we have

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{1.1.1}$$

Such a bilinear map $[\cdot,\cdot]$ is known as a **Lie Bracket**, and (1.1.1) is known as the **Jacobi Identity**.

Remark. We immediately notice that the first condition (over not just $\mathbb{C}$ but any field) implies the fact that

$$[x,y] = -[y,x] \tag{1.1.2}$$

One simply needs to apply bilinearity and the first condition to evaluate $[x+y, x+y]$. This argument reverses nicely as well, but only over fields of characteristic $\neq 2$.

One may recall that the $[\cdot,\cdot]$ notation is often used in group theory to denote the **commutator** of two elements. The reason why the same notation is used for the Lie bracket is the following.

Lemma 1.1.6. *Let A be an associative algebra. Then, the commutator map $[x,y] = xy - yx$ is a Lie bracket on A.*

Proof. Clearly, $[x,x] = xx - xx = 0$ for all $x \in A$. We now show that $[\cdot,\cdot]$ satisfies (1.1.1): for all $x, y, z \in A$, we have

$$\begin{aligned}[x,[y,z]] + [y,[z,x]] + [z,[x,y]] &= [x, yz - zy] + [y, zx - xz] + [z, xy - yx] \\ &= 6xyz - 6xyz = 0\end{aligned}$$

where we skip over some of the intermediate computations because they are tedious and uninteresting. $\square$

Lemma 1.1.6 gives us a large class of examples of Lie algebras. One of the most important of these is the following.

Example 1.1.7 (General Linear Lie Algebra). For all $n \in \mathbb{N}$, the set of all $n \times n$ matrices forms a Lie algebra under the commutator bracket: this follows immediately from applying Lemma 1.1.6 to Example 1.1.4. We call this the **General Linear Lie Algebra**, denoted $\mathfrak{gl}(n)$.

Convention. We will denote by $\mathrm{M}_n(\mathbb{C})$ the set of all $n \times n$ matrices, viewed (interchangeably) as a *set*, a *vector space* or an *associative algebra*. When viewing it as a *Lie algebra under the commutator bracket*, we will adopt the notation $\mathfrak{gl}(n, \mathbb{C})$, where $\mathbb{C}$ can be replaced by any field. We will usually abbreviate this to $\mathfrak{gl}(n)$, because we will primarily work over $\mathbb{C}$.

Lastly, we will define the notion of an abelian Lie algebra.

Definition 1.1.8 (Abelian Lie Algebra). A Lie algebra A is said to be **abelian** if for all $x \in A$, we have $[x, x] = 0$.

The reason for this terminology is that if A is an associative algebra whose multiplication map is commutative, then its commutator bracket is identically zero, making the corresponding Lie algebra abelian.

Example 1.1.9. Clearly, $\mathfrak{gl}(1)$ is abelian: for all $x, y \in \mathfrak{gl}(1) = \mathbb{C}$, we have $xy - yx = 0$.

We will now define subalgebras and homomorphisms of algebras, which will allow us to construct more examples of algebras (Lie and otherwise).

1.1.2 Subalgebras and Homomorphisms

As with objects in any category, we have subobjects and morphisms. We will define these over general algebras and apply them to get more examples of Lie algebras.

Definition 1.1.10 (Subalgebras). Let A be an algebra. A **subalgebra** of A is a subspace $B \subseteq A$ such that B is closed under the multiplication map of A. That is, for all $x, y \in B$, we have $x \cdot y \in B$.

Convention. Given an algebra A and a subset $B \subseteq A$, we will denote the statement that B is a subalgebra of A by $B \leq A$.

Definition 1.1.11 (Homomorphisms). Let A and B be algebras. A **homomorphism** $\phi : A \to B$ is a linear map that respects the multiplication maps of A and B. That is, for all $x, y \in A$, we have

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y)$$

Convention. We will define Lie subalgebras to be subalgebras with respect to the algebra structure given by the Lie bracket, and we will define Lie algebra homomorphisms to be homomorphisms that respect the Lie bracket (ie, that are algebra homomorphisms with respect to the algebra structure given by the Lie bracket).

We have the following unsurprising result.

Lemma 1.1.12. *Let A and B be algebras, and let $\phi : A \to B$ be a homomorphism. Then,*

1. $\mathrm{im}(\phi) \leq B$
2. $\ker(\phi) \leq A$

Proof. First, note that $\mathrm{im}(\phi)$ and $\ker(\phi)$ are both linear subspaces. It therefore only remains to show that they are closed under the multiplication maps of A and B.

1. Fix $x, y \in \mathrm{im}(\phi)$. Then, there exist $a, b \in A$ such that $\phi(a) = x$ and $\phi(b) = y$. Since ϕ is a homomorphism, we have

$$x \cdot y = \phi(a) \cdot \phi(b) = \phi(a \cdot b) \in \mathrm{im}(\phi)$$

so $\mathrm{im}(\phi)$ is closed under the multiplication map of B.

2. Let $x, y \in \ker(\phi)$. Then, we have

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y) = 0 \cdot 0 = 0$$

where the last equality follows from the fact that $\cdot$ is bilinear. Therefore, $x \cdot y \in \ker(\phi)$, and $\ker(\phi)$ is closed under the multiplication map of A.

□

This allows us to construct another matrix Lie algebra.

Example 1.1.13 (The Special Linear Lie Algebra). For all $n \in \mathbb{N}$, consider the trace map $\mathrm{Tr} : \mathfrak{gl}(n) \to \mathfrak{gl}(1)$. This is a (Lie) algebra homomorphism: for all $A, B \in \mathfrak{gl}(n)$,

$$\mathrm{Tr}([A, B]) = \mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0 = [\mathrm{Tr}(A), \mathrm{Tr}(B)]$$

because the Lie algebra $\mathfrak{gl}(1)$ is abelian (see Example 1.1.9). By Lemma 1.1.12, its kernel, the set of all $n \times n$ matrices of trace zero, is a Lie subalgebra of $\mathfrak{gl}(n)$. We call this the **Special Linear Lie Algebra**, denoted $\mathfrak{sl}(n)$. More explicitly,

$$\mathfrak{sl}(n) := \{A \in \mathfrak{gl}(n) \mid \mathrm{Tr}(A) = 0\}$$

and the fact that $\mathfrak{sl}(n) \leq \mathfrak{gl}(n)$ also makes it a Lie algebra in its own right.

Remark. In Example 1.1.13, we have indirectly shown that

$$\mathrm{im}([\cdot, \cdot]) = [\mathfrak{gl}(n), \mathfrak{gl}(n)] \subseteq \mathfrak{sl}(n)$$

because of the unique property of the trace that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any $A, B \in \mathfrak{gl}(n)$.

Often, when working with finite-dimensional algebras, we work with bases. As one might expect, if we can show the algebra homomorphism property for a basis, we can show it in general.

Lemma 1.1.14. *Let A and B be algebras. Let $\{e_1, \dots, e_n\}$ be a basis of A. If $\phi : A \to B$ is a linear map such that $\phi(e_i \cdot e_j) = \phi(e_i) \cdot \phi(e_j)$ for all $1 \leq i, j \leq n$, then ϕ is an algebra homomorphism.*

Proof. The result follows from bilinearity. Assume that ϕ is a linear map that preserves multiplication between basis elements. Then, for all $x, y \in A$, if we write

$$x = \sum_{k=1}^{n} \lambda_k e_k \qquad \text{and} \qquad y = \sum_{k=1}^{n} \mu_k e_k$$

for some $\lambda_k, \mu_k \in \mathbb{C}$, we have

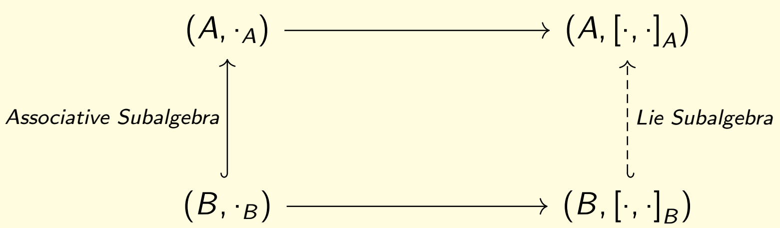
$$\phi(x \cdot y) = \phi\left(\left(\sum_{k=1}^{n} \lambda_k e_k\right) \cdot \left(\sum_{k=1}^{n} \mu_k e_k\right)\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \mu_j \phi(e_i \cdot e_j) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \mu_j \left(\phi(e_i) \cdot \phi(e_j)\right) \\
&= \left(\sum_{i=1}^{n} \lambda_i \phi(e_i) \right) \cdot \left(\sum_{j=1}^{n} \mu_j \phi(e_j) \right) \\
&= \phi(x) \cdot \phi(y)
\end{aligned}
$$

by bilinearity of $\cdot$ and linearity of ϕ, giving us the desired result. □

The very natural relationship between associative and Lie algebra structures given by Lemma 1.1.6 gives us an elegant criterion for proving that a subspace is a subalgebra of a Lie algebra whose Lie bracket is the commutator of an associative bilinear map.

Proposition 1.1.15. *Let $(A, \cdot_A)$ be an associative algebra and let $(B, \cdot_B)$ be a subalgebra of A. Denoting by $(A, [\cdot, \cdot]_A)$ the Lie algebra whose Lie bracket is the commutator of the multiplication map of A and by $(B, [\cdot, \cdot]_B)$ the Lie algebra whose Lie bracket is the commutator of the multiplication map of B, we have $(B, [\cdot, \cdot]_B) \leq (A, [\cdot, \cdot]_A)$. In other words, the following diagram commutes:*



(1.1.3)

Proof. First, observe that $[\cdot, \cdot]_B = [\cdot, \cdot]_A|_B$ (ie, the Lie bracket obtained from $\cdot_B$ agrees with the one obtained from $\cdot_A$ on B): for all $T_1, T_2 \in B$,

$$[T_1, T_2]_B = T_1 \cdot_B T_2 - T_2 \cdot_B T_1 = T_1 \cdot_A T_2 - T_2 \cdot_A T_1 = [T_1, T_2]_A$$

Therefore, since B is closed under $[\cdot, \cdot]_B$ (which, by definition, is a map from $B \times B$ to B), B must be closed under $[\cdot, \cdot]_A$. □

This allows us to construct more examples still.

Example 1.1.16 (The Upper-Triangular Lie Algebra). For $n \in \mathbb{N}$, we define the **Upper-Triangular Lie Algebra** to be the set of all $n \times n$ upper-triangular matrices (with respect to some predetermined basis), denoted $\mathfrak{t}(n)$. Given that the product of upper-triangular matrices is upper-triangular, $\mathfrak{t}(n)$ forms an associative subalgebra of $M_n(\mathbb{C})$, and therefore, a Lie subalgebra of $\mathfrak{gl}(n)$.

1.1.3 Isomorphisms

We would like to define an isomorphism of algebras to be an algebra homomorphism that is invertible, whose inverse is also an algebra homomorphism. It turns out, we can capture this information with a slightly simpler definition.

Lemma 1.1.17. *Let A and B be algebras and $\phi : A \to B$ a bijection. If, in addition, ϕ is an algebra homomorphism, then so is ϕ^{-1}.*

Proof. Assume ϕ is an algebra homomorphism. Since ϕ is linear, we know, from linear algebra, that ϕ^{-1} is linear as well. Therefore, all we need to show is that ϕ^{-1} respects the multiplication maps of A and B. To that end, fix $y_1, y_2 \in B$. We need to show that

$$\phi^{-1}(y_1 \cdot y_2) = \phi^{-1}(y_1) \cdot \phi^{-1}(y_2)$$

Since ϕ is a bijection, we know there are unique $x_1, x_2 \in A$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Indeed, $x_1 = \phi^{-1}(y_1)$ and $x_2 = \phi^{-1}(y_2)$. Then,

$$\phi^{-1}(y_1 \cdot y_2) = \phi^{-1}(\phi(x_1) \cdot \phi(x_2)) = \phi^{-1}(\phi(x_1 \cdot x_2)) = x_1 \cdot x_2 = \phi^{-1}(y_1) \cdot \phi^{-1}(y_2)$$

as required, proving that ϕ^{-1} is a Lie algebra homomorphism too. □

It is therefore not necessary to require the inverse of an algebra homomorphism to be an algebra homomorphism in order for it to be an isomorphism, though this is a fact we get for free from Lemma 1.1.17.

Definition 1.1.18 (Isomorphism of Algebras). Let A and B be algebras. An **algebra isomorphism** $\phi : A \xrightarrow{\sim} B$ is a bijective algebra homomorphism.

Over the course of this module, we will see any number of pairs of isomorphic Lie algebras. We do not offer any examples here because there will be more than enough as we go along. We will instead move onto more important—and less trivial—concepts.

1.1.4 Ideals

Throughout this subsection, we will denote by L an arbitrary Lie algebra.

Definition 1.1.19 (Ideal). We say that $I \subseteq L$ is an **ideal** of L, denoted $I \trianglelefteq L$, if I is a linear subspace of L and $[x, y] \in I$ for all $x \in L$ and $y \in I$.

Convention. We will use the notation $[L, I]$ to denote the subspace of L spanned by all elements of the form $[\ell, i]$ for $\ell \in L$ and $i \in I$.

Remark. We could equivalently require that $[I, L] \leq L$ in the definition of an ideal instead of requiring that $[x, y] \in I$ for all $x \in L$ and $y \in I$. Similarly, we can observe that it doesn't matter whether we require $[x, y] \in I$ or $[y, x] \in I$ because of (1.1.2) and bilinearity.

Example 1.1.20 (Trivial Ideals). Given any Lie algebra L, both $\{0\}$ and L are ideals of L.

In certain respects, despite their name, ideals of Lie algebras are more like normal subgroups of a group than they are like ideals of a ring.

Lemma 1.1.21. *Any ideal $I \trianglelefteq L$ is also a subalgebra of L.*

Proof. This is clear from Definition 1.1.19. □

Lemma 1.1.22. *For any Lie algebra K and homomorphism $\phi : L \to K$, we have $\ker(\phi) \trianglelefteq L$.*

Proof. From Lemma 1.1.12, we know that $\ker(\phi)$ is a linear subspace of L. We now need to show that $[x, y] \in \ker(\phi)$ for all $x \in L$ and $y \in \ker(\phi)$. To that end, fix $x \in \ker(\phi)$ and $y \in L$. Then,

$$\phi([x, y]) = [\phi(x), \phi(y)] = [0, \phi(y)] = 0$$

proving that $[x, y] \in \ker(\phi)$ as required. □

We come back to the theme of the Lie bracket being some sort of 'commutator' when we define the notion of the centre of a Lie algebra: the terminology and notation match those from group theory, where the centre consists of elements that commute with every other element of the group (making its commutator with every other element the identity).

Definition 1.1.23 (The Centre of a Lie Algebra)**.** We define the **centre** of L to be

$$Z(L) := \{x \in L \mid \forall y \in L,\ [x, y] = 0\}$$

Just as the centre of a group is a normal subgroup, so too is the centre of a Lie algebra an ideal.

Lemma 1.1.24. *$Z(L)$ is an ideal of L.*

Proof. The fact that $Z(L)$ is a subspace of L follows from the fact that $[\cdot, \cdot]$ is bilinear. Now, fix $x \in Z(L)$ and $y \in L$. Clearly, $[x, y] = 0$, and it is easily seen that $0 \in Z(L)$. □

Example 1.1.25. For all $n \in \mathbb{N}$,

$$Z(\mathfrak{gl}(n)) = \{A \in \mathfrak{gl}(n) \mid \exists \lambda \in \mathbb{C} \text{ s.t. } A = \lambda I\}$$

Proof. Let $S := \{A \in \mathfrak{gl}(n) \mid \exists \lambda \in \mathbb{C} \text{ s.t. } A = \lambda I\}$. It is clear that $S \subseteq Z(\mathfrak{gl}(n))$. Now, fix $A \in Z(\mathfrak{gl}(n))$. Then, for all $B \in \mathfrak{gl}(n)$, we have that $[A, B] = AB - BA = 0$. In particular, this implies that A commutes with all the elementary matrices E_{ij}, which are the matrices with a 1 in the ij-th position and 0 elsewhere. Therefore, A must be a diagonal matrix. □

It turns out that ideals are well-behaved under several operations.

Proposition 1.1.26 (The Behaviour of Ideals). *Let $I, J \trianglelefteq L$. Then,*

1. $I + J \trianglelefteq L$
2. $I \cap J \trianglelefteq L$
3. $[I, J] := \text{Span}(\{[i, j] \mid i \in I, j \in J\}) \trianglelefteq L$

Proof.

1. Fix $x \in I + J$. Then, there exist elements $i \in I$ and $j \in J$ such that $x = i + j$. Then, for all $y \in L$, we have
$$[x, y] = [i + j, y] = [i, y] + [j, y]$$
Since $I \trianglelefteq L$, we have $[i, y] \in I$, and since $J \trianglelefteq L$, we have $[j, y] \in J$. Therefore, $[x, y] = [i, y] + [j, y] \in I + J$.

2. Fix $x \in I \cap J$ and $y \in L$. Since $x \in I$ and $I \trianglelefteq L$, we have $[x, y] \in I$. Similarly, since $x \in J$ and $J \trianglelefteq L$, we have $[x, y] \in J$. Therefore, $[x, y] \in I \cap J$.

3. By bilinearity, the Lie bracket with some $y \in L$ of any (finite) linear combinations of elements of the form $[i, j]$ (for $i \in I$ and $j \in J$) is a linear combination of elements of the form $[[i, j], y]$. Therefore, it suffices to show that for all $i \in I$, $j \in J$, and $y \in L$, $[[i, j], y] \in [I, J]$.

   To that end, fix $i \in I$, $j \in J$ and $y \in L$. Then, applying (1.1.2) followed by the Jacobi Identity (1.1.1), we have
$$[[i, j], y] = -[y, [i, j]] = [i, [j, y]] + [j, [y, i]]$$
Since $j \in J$ and $J \trianglelefteq L$, we have $[j, y] \in L$, meaning that $[i, [j, y]] \in [I, J]$. Similarly, since $i \in I$ and $I \trianglelefteq L$, we have $[i, y] \in L$, meaning that $[j, [i, y]] = -[[i, y], j] \in [I, J]$. Therefore, $[[i, j], y] = [i, [j, y]] + [j, [y, i]] \in [I, J]$.

□

The abelian case is particularly simple.

Proposition 1.1.27 (Ideals of an Abelian Lie Algebra). *Let L be abelian. Then, every sub-vector space of L is an ideal of L.*

Proof. Let I be a sub-vector space of L. Then, for all $x \in L$ and $y \in I$, we have $[x, y] = 0$. Since I is a subspace, we must have $0 \in I$, proving that I is an ideal of L. □

We end by defining a special kind of ideal, which will become rather important.

Definition 1.1.28 (Derived Subalgebra). The **derived subalgeba** of L, denoted L', is the ideal (and subalgebra) $[L, L]$.

Note that L' is, indeed, an ideal, by the third property proven in Proposition 1.1.26.

Convention. Though L' is an ideal, we will often refer to it as either the **derived subalgebra** or the **commutator subalgebra** of L. Indeed, Lemma 1.1.21 tells us that this is a reasonable, if not the most completely descriptive, thing to do.

We will end by giving a nice technique of showing that two subspaces of L are ideals.

Proposition 1.1.29. *Let I and J be subalgebras of L be such that $I + J = L$. If $[I, I] \subseteq I$, $[J, J] \subseteq J$, and $[I, J] = \{0\}$, then both I and J are ideals of L.*

Proof. The key is that $[\cdot, \cdot]$ is bilinear. In other words, we have

$$[L, I] = [I + J, I] = [I, I] + [J, I] \subseteq I + \{0\} = I$$

Therefore, I is an ideal of L. By an identical computation, we can show that $[L, J] = J$ as well, allowing us to conclude that J, too, is an ideal of L. □

1.1.5 Quotients

We now define the notion of a quotient (Lie) algebra. For the remainder of this subsection, let L be a Lie algebra and I an arbitrary ideal of L. Given that we already have a notion of L/I—recall that I is a subspace of L, meaning we can take the quotient in a linear algebraic sense—it seems only natural to attempt to define a Lie bracket on this vector space. It turns out that the definition of an ideal allows us to do this in a very natural way.

Proposition 1.1.30. *Consider the vector space L/I. The map $[\cdot,\cdot] : L/I \times L/I \to L/I$ given by*

$$[x + I, y + I] := [x, y] + I \tag{1.1.4}$$

for all $x, y \in L$ is a Lie bracket on L/I.

Proof. We begin by showing that the Lie bracket on L/I is well-defined. Fix $x, x', y, y' \in L$ with $x - x' = i \in I$ and $y - y' = j \in I$, so that $x + I = x' + I$ and $y + I = y' + I$. Then,

$$\begin{aligned}[x, y] - [x', y'] &= [x' + i, y' + j] - [x', y'] \\ &= \cancel{[x', y']} + [i, y'] + [x', j] + [i, j] - \cancel{[x', y']} \\ &= [i, y'] + [x', j] + [i, j] \in I\end{aligned}$$

because I is an ideal, proving that $[x, y] + I = [x', y'] + I$, making the choice of representative irrelevant and the bracket on L/I well-defined.

From the definition of $[\cdot,\cdot]$ on L/I, it is clear that $[x + I, x + I] = 0$ for all $x \in L$. Now, for all $x, y, z \in L$, notice that

$$[x + I, [y + I, z + I]] = [x + I, [y, z] + I] = [x, [y, z]] + I$$

The Jacobi identity follows immediately. □

Definition 1.1.31 (Quotient Algebra). The **quotient algebra** of L with respect to I is the vector space L/I equipped with the bracket defined in (1.1.4), which we showed to be a Lie bracket in Proposition 1.1.30 above.

Example 1.1.32 (Quotienting by the Derived Subalgebra). The quotient of L by L' is always an abelian Lie algebra.

The centre is particularly well-behaved under taking quotients, a fact we will use when studying a class of Lie algebras called *nilpotent* Lie algebras.

Proposition 1.1.33. *Let* $\phi : L \twoheadrightarrow {}^{L}\big/_{Z(L)}$ *be the quotient epimorphism. Then,* $\phi(Z(L)) = Z(\phi(L)) = Z\left({}^{L}\big/_{Z(L)}\right)$.

Indeed, we can show that the map $x \mapsto x + I : L \to {}^{L}\big/_{I}$ is a Lie algebra homomorphism. More generally, we have the following results.

1.1.6 Isomorphism Theorems

Our favourite isomorphism theorems do, indeed, hold in the category of Lie algebras. Throughout this subsection, let L be a Lie algebra.

Theorem 1.1.34 (First Isomorphism Theorem). *Let* K *be a Lie algebra and* $\phi : L \to K$ *a Lie algebra homomorphism. Then,*

$$ {}^{L}\big/_{\ker(\phi)} \cong \operatorname{im}(\phi) \tag{1.1.5} $$

Theorem 1.1.35 (Second Isomorphism Theorem). *Let* $I, J \trianglelefteq L$*. Then,*

$$ {}^{I+J}\big/_{I} \cong {}^{J}\big/_{I \cap J} \tag{1.1.6} $$

We also have a correspondence between ideals of L and ideals of ${}^{L}\big/_{I}$.

Theorem 1.1.36 (The Correspondence Theorem). *Let* $I \trianglelefteq L$*. Then, there is a one-to-one correspondence between the ideals of* L *containing* I *and the ideals of* ${}^{L}\big/_{I}$*. Ie, there is a bijection*

$$ \{J \trianglelefteq L \mid J \supseteq I\} \longleftrightarrow \{J \trianglelefteq {}^{L}\big/_{I}\} \tag{1.1.7} $$

Note that each of the sets in (1.1.36) is partially ordered by inclusion.

1.1.7 Adjoints

Throughout this subsection, V will refer to a finite-dimensional vector space.

We begin with a general Lie algebra construction.

Definition 1.1.37 (General Linear Lie Algebra over an Arbitrary Vector Space)**.** We define the **General Linear Lie Algebra over** V to be the set of all linear maps from V to V, viewed as a Lie algebra under the commutator bracket on the associative algebra structure given by composition of linear maps (cf. Lemma 1.1.6). We denote it $\mathfrak{gl}(V)$.

That this is, indeed, a Lie algebra should come as no surprise. Given that this construction is well-defined over *any* vector space, we can, in particular, apply it to Lie algebras.

For the remainder of this subsection, let L denote an arbitrary Lie algebra. It turns out that we can define a rather nice map that relates L with $\mathfrak{gl}(L)$: the adjoint.

Definition 1.1.38 (Adjoint Map)**.** To every $x \in L$, we can associate the linear map

$$\mathrm{ad}(x) : L \to L : y \mapsto [x, y]$$

We call this map the **adjoint map** associated to x.

The fact that the Lie bracket is bilinear tells us that for all $x \in L$, $\mathrm{ad}(x)$ is indeed a linear map. That is, $\mathrm{ad}(x) \in \mathfrak{gl}(L)$. We can therefore view ad as a map from L to $\mathfrak{gl}(L)$ which sends any $x \in L$ to the linear map $\mathrm{ad}(x) \in \mathfrak{gl}(L)$ defined in Definition 1.1.38. Now, observe that both L and $\mathfrak{gl}(L)$ are Lie algebras. It turns out that the adjoint map is compatible this fact.

Proposition 1.1.39. *The adjoint map* $\mathrm{ad} : L \to \mathfrak{gl}(L)$ *is a Lie algebra homomorphism.*

Proof. That ad is linear follows from the fact that $[\cdot, \cdot]$ is bilinear. Now, fix $x, y \in L$, and consider the map $\mathrm{ad}([x, y]) \in \mathfrak{gl}(L)$. We need to show that

$$\mathrm{ad}([x, y]) = \mathrm{ad}(x)\,\mathrm{ad}(y) - \mathrm{ad}(y)\,\mathrm{ad}(x)$$

because the Lie bracket on $\mathfrak{gl}(L)$ is the commutator with respect to composition of linear maps.

To that end, fix $z \in L$. Then,

$$\begin{aligned}
(\mathrm{ad}(x)\,\mathrm{ad}(y) - \mathrm{ad}(y)\,\mathrm{ad}(x))(z) &= \mathrm{ad}(x)(\mathrm{ad}(y)(z)) - \mathrm{ad}(y)(\mathrm{ad}(x)(z)) \\
&= \mathrm{ad}(x)([y,z]) - \mathrm{ad}(y)([x,z]) \\
&= [x,[y,z]] - [y,[x,z]] \\
&= [x,[y,z]] + [y,[z,x]] && \text{(by (1.1.2))} \\
&= -[z,[x,y]] && \text{(by the Jacobi Identity)} \\
&= [[x,y],z] \\
&= \mathrm{ad}([x,y])(z)
\end{aligned}$$

showing that the Lie bracket in $\mathfrak{gl}(L)$ of the adjoints is indeed the adjoint of the Lie bracket in L and allowing us to conclude that ad is a homomorphism of Lie algebras. □

We also highlight the following fact.

Lemma 1.1.40. $\ker(\mathrm{ad}) = Z(L)$

Proof. This is immediate from unfolding definitions. □

Finally, we discuss a rather interesting fact about the image of the adjoint map.

Convention. Denote by $\mathrm{ad}(L)$ the set $\mathrm{im}(\mathrm{ad}) = \{\mathrm{ad}(x) \mid x \in L\}$.

We then have the following rather straightforward fact.

Lemma 1.1.41. *The set* $\mathrm{ad}(L)$ *is a Lie subalgebra of* $\mathfrak{gl}(L)$.

Proof. This is immediate from Lemma 1.1.12. □

1.1.8 Derivations

Throughout this subsection, let A be an arbitrary algebra with multiplication $\cdot$.

Definition 1.1.42. We say that a linear map $D : A \to A$ is a **derivation** if it satisfies the Leibniz rule, ie, if

$$D(x \cdot y) = x \cdot D(y) + D(x) \cdot y \tag{1.1.8}$$

for all $x, y \in A$.

Convention. We will denote the set of all derivations of an algebra A by $\mathrm{Der}(A)$.

Most readers will have encountered derivations before. We give below a classic example (over $\mathbb{R}$, for the first time so far) that the reader is sure to recognise.

Example 1.1.43. The space $C^\infty(\mathbb{R})$ of smooth $\mathbb{R} \to \mathbb{R}$ functions is an $\mathbb{R}$-algebra under pointwise addition and multiplication. The differentiation map $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) : f \mapsto f'$ is easily seen to be a derivation.

Recall that since A is a vector space, $\mathfrak{gl}(A)$ is a Lie algebra with respect to the commutator bracket (cf. Definition 1.1.37). It turns out there is a relationship between $\mathrm{Der}(A)$ and $\mathfrak{gl}(A)$.

Proposition 1.1.44. $\mathrm{Der}(A)$ *is a Lie subalgebra of* $\mathfrak{gl}(A)$*.*

Proof. That $\mathrm{Der}(A)$ is a subspace of $\mathfrak{gl}(A)$ is not too difficult to show: it is clear that the zero map satisfies (1.1.8), and it readily follows from the bilinearity of $\cdot$ that $\mathrm{Der}(A)$ is closed under addition and scalar multiplication.

We now need to show that $\mathrm{Der}(A)$ is closed under the commutator bracket. Fix $D, E \in \mathrm{Der}(A)$. We need to show that $[D, E] = DE - ED$ satisfies (1.1.8). Indeed, for all $x, y \in A$,

$$\begin{aligned}(DE - ED)(x \cdot y) &= D(E(x \cdot y)) - E(D(x \cdot y)) \\ &= D(x \cdot E(y) + E(x) \cdot y) - E(x \cdot D(y) + D(x) \cdot y)\end{aligned}$$

which can be simplified, if tediously, to the desired form. □

Differentiation of smooth functions is not the only example of a derivation. In fact, in the context of Lie algebras, we have the following rather interesting fact.

Proposition 1.1.45. *For all $x \in L$, the adjoint map $\mathrm{ad}(x) : L \to L : y \mapsto [x, y]$ associated with x is a derivation.*

Proof. We already know that $\mathrm{ad}(x) \in \mathfrak{gl}(L)$. It only remains to show that $\mathrm{ad}(x)$ satisfies (1.1.8) with respect to $[\cdot, \cdot]$. To that end, fix $y, z \in L$. Then, we have that

$$\begin{aligned} \mathrm{ad}(x)([y, z]) &= [x, [y, z]] \\ &= -[y, [z, x]] - [z, [x, y]] \\ &= [y, [x, z]] + [[x, y], z] \\ &= [y, \mathrm{ad}(x)(z)] + [\mathrm{ad}(x)(y), z] \end{aligned}$$

as required. □

We therefore have the following chain of Lie algebra inclusions.

Lemma 1.1.46. $\mathrm{ad}(L) \leq \mathrm{Der}(L) \leq \mathfrak{gl}(L)$.

Proof. We only need to show that $\mathrm{ad}(L) \leq \mathrm{Der}(L)$, because we have already shown that $\mathrm{Der}(L) \leq \mathfrak{gl}(L)$ in Proposition 1.1.44. By Proposition 1.1.45, we know $\mathrm{ad}(L) \subseteq \mathrm{Der}(L)$. We know that $\mathrm{ad}(L)$ is a linear subspace of $\mathrm{Der}(L)$ because it is a linear subspace of $\mathfrak{gl}(L)$ that is contained in $\mathrm{Der}(L)$. For the same reason, it is also a Lie subalgebra of $\mathrm{Der}(L)$, because we know that the commutator of two adjoints is the adjoint of some Lie bracket in L (cf. Proposition 1.1.39). In particular, it is contained in $\mathrm{Der}(L)$ because it is the adjoint of some element of L (by Proposition 1.1.45). Therefore, $\mathrm{ad}(L)$ is a Lie subalgebra of $\mathrm{Der}(L)$. □

Lemma 1.1.46 is hardly surprising, given the facts we have already proven. What is a lot less obvious, though, is that $\mathrm{ad}(L)$ is more than just a subalgebra of $\mathrm{Der}(L)$: it is an ideal. We begin by proving a cute fact.

Lemma 1.1.47. *For all* $\ell \in L$ *and* $D \in \mathrm{Der}(L)$*,*

$$[D, \mathrm{ad}(\ell)] = \mathrm{ad}(D(\ell))$$

Proof. Fix $\ell \in L$ and $D \in \mathrm{Der}(L)$. For all $x \in L$, bearing in mind that the $[\cdot, \cdot]$ notation here is used for both the commutator bracket in $\mathfrak{gl}(L)$ and the Lie bracket in L,

$$\begin{aligned}
[D, \mathrm{ad}(\ell)](x) &= (D \circ \mathrm{ad}(\ell) - \mathrm{ad}(\ell) \circ D)(x) \\
&= D(\mathrm{ad}(\ell)(x)) - \mathrm{ad}(\ell)(D(x)) \\
&= D([\ell, x]) - [\ell, D(x)] \\
&= [\ell, D(x)] + [D(\ell), x] - [\ell, D(x)] \\
&= [D(\ell), x]
\end{aligned}$$

as required. □

It follows quite readily that $\mathrm{ad}(L) \trianglelefteq \mathrm{Der}(L)$.

Corollary 1.1.48. $\mathrm{ad}(L)$ *is an ideal of* $\mathrm{Der}(L)$ *in* $\mathfrak{gl}(L)$.

Proof. Lemma 1.1.47 clearly tells us the commutator of a derivation and an adjunction is an adjunction, which is exactly what it means for $\mathrm{ad}(L)$ to be an ideal of $\mathrm{Der}(L)$ in $\mathfrak{gl}(L)$. □

The containment of $\mathrm{ad}(L)$ in $\mathrm{Der}(L)$ can be proper, though this is not necessary.

Example 1.1.49 (Adjunctions and Derivations in Abelian Lie Algebras)**.** Let L be an abelian Lie algebra. Then, any linear map $T \in \mathfrak{gl}(L)$ is a derivation: for all $x, y \in L$,

$$T([x, y]) = T(0) = 0 = 0 + 0 = [T(x), y] + [x, T(y)]$$

Furthermore, since $\mathrm{ad}(L) = \{0\}$ (because L is abelian), if T is any non-zero linear map—for example, $T = \mathrm{id}_L$—then T is a derivation that is not an adjunction, making the inclusion of $\mathrm{ad}(L)$ into $\mathrm{Der}(L)$ strict.

1.1.9 Structure Constants

Fix $n \in \mathbb{N}$, and let L be an n-dimensional Lie algebra. Consider some $\mathbb{C}$-basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of L. Given the fundamentally linear algebraic nature of Lie algebras, it is natural to study what happens when we apply the Lie bracket to elements of $\mathcal{B}$.

Definition 1.1.50 (Structure Constants). Fix $i, j \in \{1, \ldots, n\}$. We know that there exist unique constants $s_{ij1}, s_{ij2}, \ldots, s_{ijn}$ such that

$$[e_i, e_j] = \sum_{k=1}^{n} s_{ijk} e_k$$

We call the scalars $\{s_{ijk}\}_{1 \leq i,j,k \leq n}$ the **structure constants** of L with respect to $\mathcal{B}$.

In other words, the structure constant s_{ijk} of L with respect to $\mathcal{B}$ is the kth coordinate (with respect to $\mathcal{B}$) of the Lie bracket of the ith and jth elements of $\mathcal{B}$.

As one might expect from their name, the structure constants of a Lie algebra define it uniquely up to isomorphism.

Theorem 1.1.51 (The Meaning of Structure Constants). *Let L_1 and L_2 be n-dimensional Lie algebras over $\mathbb{C}$. Then, $L_1 \cong L_2$ if and only if there exist $\mathbb{C}$-bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of L_1 and L_2 respectively such that the structure constants of L_1 with respect to $\mathcal{B}_1$ are equal to those of L_2 with respect to $\mathcal{B}_2$.*

Proof. First, note that the fact that L_1 and L_2 have the same dimension does not, in general, make them isomorphic as Lie algebras. They are certainly isomorphic as $\mathbb{C}$-vector spaces, but this is not enough. For example, as we shall see later on (Section 1.2.2), there are two non-isomorphic Lie algebras of dimension 2 over $\mathbb{C}$. With this in mind, we proceed with the proof.

($\implies$) Assume that $L_1 \cong L_2$ via some isomorphism $\phi : L_1 \to L_2$. Let $\mathcal{B}_1 = \{e_1, \ldots, e_n\}$ be any basis of L_1. We know that the set

$$\mathcal{B}_2 = \{f_i = \phi(e_i) \mid 1 \leq i \leq n\}$$

is a basis of L_2. What's more, for all $1 \leq i, j \leq n$, since ϕ is an isomorphism of Lie algebras,

$$\phi([e_i, e_j]) = [\phi(e_i), \phi(e_j)] = [f_i, f_j]$$

Therefore, denoting by $\{s_{ijk}\}_{1 \leq i,j,k \leq n}$ the structure constants of L_1 with respect to $\mathcal{B}_1$,

$$[f_i, f_j] = \phi([e_i, e_j]) = \phi\left(\sum_{k=1}^{n} s_{ijk} e_k\right) = \sum_{k=1}^{n} s_{ijk} \phi(e_k) = \sum_{k=1}^{n} s_{ijk} f_k$$

By uniqueness of structure constants (ie, of coordinates), it must be that the structure constants of L_2 with respect to $\mathcal{B}_2$ are also $\{s_{ijk}\}_{1 \leq i,j,k \leq n}$.

($\Longleftarrow$) Assume that there exist bases $\mathcal{B}_1 = \{e_1, \ldots, e_n\}$ and $\mathcal{B}_2 = \{f_1, \ldots, f_n\}$ of L_1 and L_2 respectively such tha the structure constants of L_1 with respect to B_1 are equal to those of L_2 with respect to $\mathcal{B}_2$. Denote these by $\{s_{ijk}\}_{1 \leq i,j,k \leq n}$. Define the linear isomorphism $\phi : L_1 \xrightarrow{\sim} L_2$ that maps e_k to f_k for all $1 \leq k \leq n$. We show that ϕ is, in fact, a Lie algebra isomorphism.

We only need to show that ϕ is a Lie algebra homomorphism, ie, that it preserves Lie brackets. Since L_1 and L_2 have the same structure constants with respect to $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, we can see that for all $1 \leq i, j \leq n$,

$$\phi([e_i, e_j]) = \phi\left(\sum_{k=1}^{n} s_{ijk} e_k\right) = \sum_{k=1}^{n} s_{ijk} \phi(e_k) = \sum_{k=1}^{n} s_{ijk} f_k = [f_i, f_j] = [\phi(e_i), \phi(e_j)]$$

Lemma 1.1.14 then allows us to conclude that ϕ preserves the Lie bracket for *all* elements, not just basis elements. This makes it a Lie algebra homomorphism in addition to being a linear isomorphism, making it a Lie algebra isomorphism as desired.

□

In a sense, structure constants give us a way of describing Lie algebraic phenomena numerically in the same way that coordinates give us a way of describing linear algebraic phenomena numerically. Indeed, both notions are basis-dependent. In fact, from the very definition of a Lie algebra, we can glean some information as to the behaviour of structure constants.

For the remainder of this subsection, just as we did at the beginning of this subsection, fix a Lie algebra L, a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, and structure constants $\{s_{ijk}\}_{1 \leq i,j,k \leq n}$ of L with respect to $\mathcal{B}$.

Lemma 1.1.52. *For all* $1 \leq i, j, k \leq n$,

1. $s_{iik} = 0$.
2. $s_{ijk} = -s_{jik}$.

Proof. We prove the first point using the fact that the Lie bracket of any element with itself is 0. The proof of the second point is nearly identical, except it uses antisymmetry (1.1.2) instead.

Fix $1 \leq i, k \leq n$. From the definition of the Lie bracket, we know that $[e_i, e_i] = 0$. Therefore, since s_{iik} is the kth coordinate of $[e_i, e_i]$ with respect to $\mathcal{B}$, s_{iik} must be 0.

The proof of the second point is analogous. □

We can also describe the Jacobi Identity using structure constants, but this is significantly more cumbersome. Therefore, we do not do it here.

Example 1.1.53 (Computing Structure Constants in $\mathbb{R}^3$)**.** The Euclidean space $\mathbb{R}^3$ can be viewed as a Lie algebra under the cross product: it is straightforward to show that the axioms of a Lie algebra are, indeed, satisfied. We can therefore compute the structure constants $\{s_{ijk}\}_{1 \leq i,j,k \leq 3}$ of $\mathbb{R}^3$ under the cross product with respect to the standard basis $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$.

There are 27 structure constants in total, but we can use Lemma 1.1.52 to simplify things. Immediately, we can see that 9 of them are zero:

$$s_{111} = s_{112} = s_{113} = s_{221} = s_{222} = s_{223} = s_{331} = s_{332} = s_{333} = 0$$

Furthermore, of the remaining 18, 9 are the negatives of the other 9. Therefore, we only really need to compute 9 structure constants to know the remaining 27. Since each structure constant is a coordinate of a vector, we only need to compute 3 vectors, namely, the cross products of the basis vectors with each other.

Observe that

$$e_1 \times e_2 = e_3 \qquad e_2 \times e_3 = e_1 \qquad e_3 \times e_1 = e_2$$

This tells us that

$$\begin{aligned} s_{121} &= 0 & s_{122} &= 0 & s_{123} &= 1 \\ s_{231} &= 1 & s_{232} &= 0 & s_{233} &= 0 \\ s_{311} &= 0 & s_{312} &= 1 & s_{313} &= 0 \end{aligned}$$

Combining this with Lemma 1.1.52, we can see that the structure constants of $\mathbb{R}^3$ with respect to the standard basis are

$$\begin{aligned} s_{111} &= 0 & s_{112} &= 0 & s_{113} &= 0 \\ s_{121} &= 0 & s_{122} &= 0 & s_{123} &= 1 \\ s_{131} &= 0 & s_{132} &= -1 & s_{133} &= 0 \\ s_{211} &= 0 & s_{212} &= 0 & s_{213} &= -1 \\ s_{221} &= 0 & s_{222} &= 0 & s_{223} &= 0 \\ s_{231} &= 1 & s_{232} &= 0 & s_{233} &= 0 \\ s_{311} &= 0 & s_{312} &= 1 & s_{313} &= 0 \\ s_{321} &= -1 & s_{322} &= 0 & s_{323} &= 0 \\ s_{331} &= 0 & s_{332} &= 0 & s_{333} &= 0 \end{aligned}$$

1.1.10 Direct Sums

In this subsection, we briefly describe the theory of the direct sum of two Lie algebras. Let L_1 and L_2 be arbitrary Lie algebras. Just as we did in Proposition 1.1.30, we will define a Lie bracket on the vector space $L_1 \oplus L_2$, and define the Lie algebra direct sum of L_1 and L_2 to be this vector space equipped with this bracket.

Proposition 1.1.54. *Define the map* $[\cdot,\cdot] : (L_1 \oplus L_2) \times (L_1 \oplus L_2) \to (L_1 \oplus L_2)$ *given by*

$$[x_1 \oplus x_2, y_1 \oplus y_2] := [x_1, y_1] \oplus [x_2, y_2] \tag{1.1.9}$$

for all $x_1, y_1 \in L_1$ *and* $x_2, y_2 \in L_2$*. Then,* $[\cdot,\cdot]$ *is a Lie bracket on* $L_1 \oplus L_2$*.*

Proof. First, note that the Lie bracket $[\cdot,\cdot]$ is bilinear because it is bilinear on each component. Now, fix $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$. Then, we have

$$[x_1 \oplus x_2, x_1 \oplus x_2] = [x_1, x_1] \oplus [x_2, x_2] = 0 \oplus 0 = 0$$

proving that $[\cdot,\cdot]$ satisfies the first property of a Lie bracket. Finally, for all $z_1 \in L_1$ and $z_2 \in L_2$, observe that

$$\begin{aligned}[x_1 \oplus x_2, [y_1 \oplus y_2, z_1 \oplus z_2]] &= [x_1 \oplus x_2, [y_1, z_1] \oplus [y_2, z_2]] \\ &= [x_1, [y_1, z_1]] \oplus [x_2, [y_2, z_2]]\end{aligned}$$

and similarly for the other terms in the Jacobi identity. Since the Jacobi identity holds in L_1 and L_2, it must hold in $L_1 \oplus L_2$. Therefore, $[\cdot,\cdot]$ is a Lie bracket on $L_1 \oplus L_2$. □

Definition 1.1.55 (Direct Sum)**.** The **direct sum** of L_1 and L_2 is the vector space $L_1 \oplus L_2$ equipped with the bracket defined in (1.1.9), which we showed to be a Lie bracket in Proposition 1.1.54 above.

We can repeat this definition successively to define the direct sum of any finite number of Lie algebras. We will not explore this idea in any more detail and will take it for granted.

After all of this important theory, we are finally ready to study concrete Lie algebras and their properties. Given that the objective of this module is to classify all semi-simple Lie algebras over $\mathbb{C}$, a natural place to begin is classifying *all* Lie algebras of small dimension. We will do this in the next section.

1.2 Lie Algebras of Dimension ≤ 3

It turns out that we do not need any particularly sophisticated machinery to classify all Lie algebras of dimension less than or equal to 3.

1.2.1 Abelian Lie Algebras and Lie Algebras of Dimension 1

We begin with a simple observation about abelian Lie algebras.

Proposition 1.2.1. *Fix $n \in \mathbb{N}$. Then, any abelian Lie algebra of dimension n is isomorphic to $\mathbb{C}^n$ with the zero bracket.*

Proof. Let L be a Lie algebra of dimension n. We know there exists a $\mathbb{C}$-linear isomorphism $\phi : L \to \mathbb{C}^n$. It follows immediately that for any $x, y \in L$,

$$\phi([x, y]) = \phi(0) = 0 = [\phi(x), \phi(y)]$$

A similar argument will show that $\phi^{-1} : \mathbb{C}^n \to L$, viewed as a linear map, is a Lie algebra homomorphism as well, proving that $L \cong \mathbb{C}^n$. □

The classification of Lie algebras in 1 dimension is then straightforward. We will begin by a rather strong but straightforward result on one-dimensional subspaces of Lie algebras.

Proposition 1.2.2. *Let L be a Lie algebra. Any 1-dimensional subspace of L is an abelian Lie subalgebra.*

Proof. Let K be a sub-vector space of dimension 1. We know any $\mathbb{C}$-basis of K consists of a single, nonzero element. Consider such a basis element x. For any $y_1, y_2 \in L$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $y_1 = \lambda_1 x$ and $y_2 = \lambda_2 x$. Then,

$$[y_1, y_2] = [\lambda_1 x_1, \lambda_1 x_2] = \lambda_1 \lambda_2 [x, x] = 0$$

proving that $[\cdot, \cdot] = 0$. Since K is a subspace, $0 \in K$, proving that K is a Lie subalgebra. □

The classification of Lie algebras of dimension 1 is then immediate.

Corollary 1.2.3. *Any Lie algebra of dimension* 1 *is abelian, isomorphic to* $\mathbb{C}$ *equipped with the zero bracket.*

Proof. Let L be a Lie algebra of dimension 1. That L is abelian follows from applying Proposition 1.2.2 to L viewed as a subspace of itself. The isomorphism then follows immediately from Proposition 1.2.1. □

We can now turn our attention to the slightly more non-trivial problem of classifying non-abelian Lie algebras of dimension 2 and 3.

1.2.2 Lie Algebras of Dimension 2

From Proposition 1.2.1, we already know that there is only one abelian Lie algebra of dimension 2. The question remains, how many non-abelian Lie algebras of dimension 2 are there?

We begin by giving an example.

Example 1.2.4 (A Two-Dimensional Non-Abelian Lie Algebra)**.** Consider the set

$$\mathfrak{r}_2 := \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \;\middle|\; a, b \in \mathbb{C} \right\} = \mathrm{Span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \subseteq \mathfrak{gl}(2)$$

Clearly, $\mathfrak{r}_2$ is a linear subspace of $\mathfrak{gl}(2)$. Furthermore, One can show that

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

proving that $\mathfrak{r}_2$ is closed under the commutator bracket. It follows that $\mathfrak{r}_2$ is a Lie subalgebra of $\mathfrak{gl}(2)$, and therefore, a 2-dimensional Lie algebra in its own right.

The reason we are interested in the above example will become clear, and we will reserve the notation $\mathfrak{r}_2$ for this particular Lie algebra. For the remainder of this section, denote by L an arbitrary non-abelian Lie subalgebra of dimension 2.

We will begin by describing the derived subalgebra L' (cf. Definition 1.1.28) of L.

Lemma 1.2.5. *For any $\mathbb{C}$-basis $\{u, v\}$ of L, we have that $L' = \mathrm{Span}([u, v])$.*

Proof. Let $\{u, v\}$ be a basis of L. Define $x := [u, v]$. Since L is non-abelian, $x \neq 0$, making $X := \mathrm{Span}(x)$ a 1-dimensional subspace of L. Seeing as $L' = [L, L] = \mathrm{Span}(\{[x, y] \mid x, y \in L\})$, it is clear that $L' \supseteq X$. It remains to show that $L' \subseteq X$.

It suffices to show that $\{[x, y] \mid x, y \in L\} \subseteq X$. To that end, fix $a, b \in L$. We know there exist $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{C}$ such that $a = \lambda_1 u + \mu_1 v$ and $b = \lambda_2 u + \mu_2 v$. Then,

$$\begin{aligned}[a, b] &= [\lambda_1 u + \mu_1 v, \lambda_2 u + \mu_2 v] \\ &= \lambda_1\lambda_2 \underbrace{[u, u]}_{=0} + \lambda_1\mu_2 [u, v] + \mu_1\lambda_2 [v, u] + \mu_1\mu_2 \underbrace{[v, v]}_{=0} \\ &= (\lambda_1\mu_2 - \mu_1\lambda_2) [u, v] \in X\end{aligned}$$

as required. □

This tells us, in particular, that the span of the commutator of any basis of L is an ideal. We now have everything we need to describe L.

Proposition 1.2.6. *L is isomorphic to $\mathfrak{r}_2$.*

Proof. It suffices to show that L admits a basis $\{x, y\}$ such that $[x, y] = y$, as this will readily yield the right structure constants.[1]

Let $\{u, v\}$ be an arbitrary $\mathbb{C}$-basis of L. Let $y := [u, v]$. Since L is non-abelian, $y \neq 0$. Therefore, there exists some $z \in L \setminus \{0\}$ that is linearly independent of y. Since $\mathrm{Span}(y) = L' \trianglelefteq L$, we know that $[z, y] \in L'$. In particular, $\exists \lambda \in \mathbb{C}$ such that $[z, y] = \lambda y$. Furthermore, since y and z are linearly independent and L is non-abelian, $\lambda \neq 0$. So, define $x := \lambda^{-1} z$. Then, x is still linearly independent of y, making $\{x, y\}$ a basis of L, and $[x, y] = y$, as required. □

Yes, it's true! Up to isomorphism, there is only one non-abelian Lie algebra of dimension 2.

[1]Alternatively, if we can show that $[x, y] = y$, it will follow immediately that the linear isomorphism sending x to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and y to $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is, indeed, a Lie algebra isomorphism.

Therefore, there are only two Lie algebras of dimension 2: one non-abelian one and one abelian one.

We can now turn our attention to the classification of Lie algebras in dimension 3.

1.2.3 Lie Algebras of Dimension 3

We already know the isomorphism class of all abelian Lie algebras of dimension 3 from Proposition 1.2.1. The question remains, how many non-abelian Lie algebras of dimension 3 are there?

We begin with a simple observation. Let L be a Lie algebra of dimension 3. Assume that L is non-abelian. Then, we know that $\mathrm{Z}(L) \neq L$ and that $L' = [L, L] \neq 0$. We can then conclude that $\dim(\mathrm{Z}(L)) \in \{0, 1, 2\}$ and $\dim(L') \in \{1, 2, 3\}$. The strategy we use to determine the isomorphism class of L will be to consider all possible dimensions of L' and use our understanding of $\mathrm{Z}(L)$ to distinguish between isomorphism classes given a dimension of L'.

The case where $\dim(L') = 1$.

We have two possibilities: either $L' \subseteq \mathrm{Z}(L)$ or $L' \nsubseteq \mathrm{Z}(L)$. We will consider both cases separately. We begin with an example of a 3-dimensional Lie algebra L such that $\dim(L') = 1$ and $L' \subseteq \mathrm{Z}(L)$.

Example 1.2.7 (The Heisenberg Lie Algebra of Dimension 3)**.** Define the **Heisenberg Lie Algebra** $\mathfrak{u}(n)$ (for $n \in \mathbb{N}$) to be all matrices of the form

$$\begin{bmatrix} 0 & & * \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{bmatrix}$$

That is, $\mathfrak{u}(n)$ is the set of all upper-triangular matrices in $\mathfrak{gl}(n)$ with zeroes on the diagonal. One can apply Proposition 1.1.15 to show that $\mathfrak{u}(n)$ is a Lie subalgebra of $\mathfrak{gl}(n)$ because it is an associative subalgebra of $\mathfrak{gl}(n)$, which one can show by showing it is an associative subalgebra of $\mathfrak{t}(n)$ (the diagonal of a product of upper-triangular matrices is the componentwise product of the diagonals, so when the diagonals are zero, so is that of the product).

We now consider the specific case of $n = 3$. Denote by E_{ij} the 3×3 matrix whose ij entry is 1 and whose other entries are all 0. It is easily seen that $\{E_{12}, E_{13}, E_{23}\}$ is a basis of $\mathfrak{u}(3)$. Therefore, we can, indeed, see that $\mathfrak{u}(3)$ is 3-dimensional. Indeed, if $L = \mathfrak{u}(3)$, then we can show that $\dim(L') = 1$ and $L' \subseteq Z(L)$.

To show that $\dim(L') = 1$, we can compute the commutator of two matrices in $\mathfrak{u}(3)$. Define

$$X = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, $X, Y \in \mathfrak{u}(3)$. Then, computing their commutator,

$$\begin{aligned} [X,Y] &= \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & af \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & dc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & af - dc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The span of every such commutator is easily seen to be the one-dimensional subspace spanned by E_{13}. Indeed, $E_{13} = [E_{12}, E_{21}] \in [L, L] = L'$. Therefore, $\dim(L') = 1$.

Since we have computed L' explicitly, to show that $L' \subseteq Z(L)$, we only need to show that E_{13} commutes with every element of L. Indeed, for an arbitrary $X \in \mathfrak{u}(3)$ as defined above,

$$[X, E_{13}] = \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} = 0 - 0 = 0$$

proving that $L' \subseteq Z(L)$.

It turns out that up to isomorphism, there is no other Lie algebra of dimension 3 with $\dim(L') = 1$

and $L' \subseteq Z(L)$.

Theorem 1.2.8. *Let L be a Lie algebra of dimension 3 such that $\dim(L') = 1$ and $L' \subseteq Z(L)$. Then, L is isomorphic to the Heisenberg Lie algebra $\mathfrak{u}(3)$.*

Proof. Since $\dim(L') = 1$, there exists some $z \in L'$ such that $L' = \text{Span}(z)$. Indeed, since $z \in L' = [L, L]$, we can conclude that there exist $f, g \in L$ such that $z = [f, g]$. Our strategy is to show that $\{f, g, z\}$ is a basis of L. Then, the linear isomorphism from L to $\mathfrak{u}(3)$ identifying f with E_{12}, g with E_{21} and z with E_{13} will be a Lie algebra isomorphism, proving the theorem.

To show that $\{f, g, z\}$ is a basis of L, we only need to show that f, g and z are linearly independent. To that end, fix constants $\alpha, \beta, \gamma \in \mathbb{C}$ and assume that

$$\alpha f + \beta g + \gamma z = 0 \tag{1.2.1}$$

Since we assumed that $L' \subseteq Z(L)$, we know that $z \in Z(L)$. That is, its Lie bracket with any element of L is 0. We can take advantage of this to show that $\alpha = \beta = 0$, forcing $\gamma = 0$ as well. To show that $\alpha = 0$, we bracket (1.2.1) with g:

$$\alpha\,[f, g] + \beta\,[g, g] + \gamma\,[z, g] = 0 \iff \alpha z = 0 \iff z = 0$$

because $z \neq 0$. In similar fashion, if we bracket (1.2.1) with f, we can show that $\beta = 0$. This forces $\gamma = 0$ because again, $z \neq 0$, proving that $\{f, g, z\}$ is linearly independent. Therefore, $\{f, g, z\}$ is a basis of L, and we can conclude that L is isomorphic to $\mathfrak{u}(3)$. □

Now that we have classified all 3-dimensional Lie algebras L with $\dim(L') = 1$ and $L' \subseteq Z(L)$, we can turn our attention to the case where $L' \not\subseteq Z(L)$. We will begin with an example.

Example 1.2.9. Let $L = \mathfrak{t}(2)$ be the set of 2×2 upper-triangular matrices (cf. Example 1.1.16) with basis $\{E_{11}, E_{12}, E_{22}\}$ of matrices E_{ij} with 1s in the ij entries and 0s everywhere else. It turns out that $L \cong \mathfrak{r}_2 \oplus \mathfrak{gl}(1)$, where $\mathfrak{gl}(1)$ is $\mathbb{C}$ under the commutator bracket and $\mathfrak{r}_2$ is as defined in Example 1.2.4: one can show that the linear isomorphism mapping E_{11} to $(E_{11}, 0)$, E_{12} to $(E_{12}, 0)$ and E_{22} to $(0, 1)$ is invariant under the Lie bracket. Furthermore, one can show that sorry

1.3 Solvability and Nilpotency

We now begin discussing some nontrivial objects in the theory of Lie algebras. Throughout this section, L will denote an arbitrary Lie algebra.

1.3.1 Descending Series of Ideals

We begin by defining the so-called **derived series** that consists of repeated derivations, that is, repeated performing of the $'$ operation on a Lie algebra (that is, repeatedly computing the derived subalgebra).

Definition 1.3.1 (Derived Series). The **derived series** of L is the descending series of ideals

$$L = L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$$

where $L^{(i)} := \left[L^{(i-1)}, L^{(i-1)}\right]$ for $i \geq 1$.

Each $L^{(i)}$ is nothing but the derived subalgebra of $L^{(i-1)}$.

We have a special term for Lie algebras for which the derived series stabilises at 0.

Definition 1.3.2 (Solvability). L is said to be **solvable** if there exists an $n \in \mathbb{N}$ such that $L^{(n)} = 0$.

We have already encountered a trivial family of solvable Lie algebras.

Example 1.3.3. Every abelian Lie algebra L is solvable. Its derived series is simply

$$L = L^{(0)} \supseteq L^{(1)} = [L, L] = 0$$

There is also a less trivial example.

Example 1.3.4. $\mathfrak{r}_2$ (cf. Example 1.2.4) is solvable. Its derived series is

$$\begin{aligned} \mathfrak{r}_2 = \mathfrak{r}_2^{(0)} &= \operatorname{Span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \\ &\supsetneq \mathfrak{r}_2^{(1)} = [\mathfrak{r}_2, \mathfrak{r}_2] = \operatorname{Span}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \\ &\supsetneq \mathfrak{r}_2^{(2)} = \left[\mathfrak{r}_2^{(1)}, \mathfrak{r}_2^{(1)}\right] = 0 \end{aligned}$$

because $\mathfrak{r}_2^{(1)}$, being one-dimensional, is abelian.

Given that we understand all Lie algebras of dimension 2, we can make such seemingly sweeping statements as "All Lie algebras of dimension 2 are solvable."

Next, we define the **lower central series** of a Lie algebra. This is a series of ideals that is similar to the derived series, but instead of repeatedly taking the derived subalgebra, we repeatedly take the commutator with the parent Lie algebra.

Definition 1.3.5 (Lower Central Series)**.** The **lower central series** of L is the descending series of ideals

$$L = L^0 \supseteq L^1 \supseteq L^2 \supseteq \cdots$$

where $L^i := \left[L, L^{i-1}\right]$ for $i \geq 1$.

The use of parentheses when describing the derived series is deliberate: notice that we have dropped the parentheses in the above definition.

Convention. Elements of the derived series are denoted $L^{(i)}$, with parenthesised superscript indices, whereas elements of the lower central series are denoted L^i, with no parentheses around the indices.

We give a special term to Lie algebras where this process of repeatedly taking the commutator with the parent Lie algebra stabilises at 0.

Definition 1.3.6 (Nilpotency). L is said to be **nilpotent** if there exists an $n \in \mathbb{N}$ such that $L^n = 0$.

The name is not accidental: in nilpotent Lie algebras, all adjoint maps are nilpotent. We will prove this, and even more interesting facts, in due course.

We have already encountered a trivial family of nilpotent Lie algebras.

Example 1.3.7. Every abelian Lie algebra L is nilpotent. Its lower central series is simply

$$L = L^0 \supseteq L^1 = [L, L] = 0$$

There are nontrivial examples as well, but it will be easier to construct them once we develop more machinery.

There is a very important relationship between the derived series and the lower central series.

Lemma 1.3.8. *For all $i \in \mathbb{N}$, $L^i \supseteq L^{(i)}$.*

Proof. We argue by induction on i. The base case is trivial, because $L^0 = L = L^{(0)}$. Now, fix $i \in \mathbb{N}$ and assume that $L^i \supseteq L^{(i)}$. Then,

$$\begin{aligned} L^{i+1} = \left[L, L^i\right] &= \left[L, L^{(i)}\right] \\ &= \operatorname{Span}\left(\left\{[\ell, x] \;\middle|\; x \in L^{(i)}, \ell \in L\right\}\right) \\ &\supseteq \operatorname{Span}\left(\left\{[\ell, x] \;\middle|\; x \in L^{(i)}, \ell \in L^{(i)}\right\}\right) \\ &= \left[L^{(i)}, L^{(i)}\right] = L^{(i+1)} \end{aligned}$$

where the inclusion on the third line follows from the fact that $L^{(i)} \subseteq L$. This completes the induction and proves the desired result for all $i \in \mathbb{N}$. $\square$

This gives us a natural relationship between nilpotency and solvability.

Corollary 1.3.9. *If L is nilpotent, then L is solvable.*

Proof. Lemma 1.3.8 tells us that for all $n \in \mathbb{N}$, $L^n = 0$ implies $L^{(n)} = 0$. Thus, if such an n exists that makes L nilpotent, the same n would also make L solvable. □

The converse is not true.

Counterexample 1.3.10 (A Lie algebra that is solvable but not nilpotent). In Example 1.3.4, we saw that $\mathfrak{r}_2$ is solvable. However, $\mathfrak{r}_2$ is not nilpotent: its lower central series stabilises at a nonzero ideal. Explicitly,

$$\begin{aligned}
\mathfrak{r}_2 = \mathfrak{r}_2^0 &= \operatorname{Span}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)\\
&\supsetneq \mathfrak{r}_2^1 = [\mathfrak{r}_2, \mathfrak{r}_2] = \operatorname{Span}\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)\\
&= \mathfrak{r}_2^2 = \left[\mathfrak{r}_2, \mathfrak{r}_2^1\right] = \operatorname{Span}\left(\left[\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right], \left[\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right]\right) = \operatorname{Span}\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)\\
&= \mathfrak{r}_2^3 = \mathfrak{r}_2^4 = \cdots\\
&\neq 0
\end{aligned}$$

In fact, one can construct several more counterexamples. Once we develop more machinery, we will be able to show, among other things, that every $\mathfrak{t}(n)$ is solvable but not nilpotent.

For now, we end this subsection with a fact about the centres of nonzero, nilpotent Lie algebras. The point of the following is that the converse tells us when a nonzero Lie algebra is *not* nilpotent, allowing us to construct counterexamples to the converse of Corollary 1.3.9.

Lemma 1.3.11. *If L is nonzero and nilpotent, its centre is nonzero.*

Proof. Let n be the largest natural number such that $L^n \neq 0$. That is, let $n \in \mathbb{N}$ be such that

$$L = L^0 \supseteq L^1 \supseteq \cdots \supseteq L^n \supsetneq L^{n+1} = 0$$

We know such an n exists not only because L is nilpotent but also because $L \neq 0$, meaning that the lower central series cannot be trivial (ie, it consists of at least one proper inclusion—namely, that of the zero ideal with a non-zero ideal). By definition, $[L, L^n] = L^{n+1} = 0$. Hence, $L^n = Z(L)$. And, as discussed above, $L^n \neq 0$. Therefore, $Z(L) \neq 0$, as required. □

We now develop some machinery that allows us to construct solvable and nilpotent Lie *sub*algebras (that would, in particular, imply solvability and nilpotency when we view these Lie subalgebras as Lie algebras in their own right).

1.3.2 Ideals, Quotients and Subalgebras

For a subalgebra to be solvable means exactly what one would imagine.

Definition 1.3.12 (Solvability of Subalgebras). We say a subalgebra of L is **solvable** if it is solvable as a Lie algebra in its own right.

For the remainder of this subsection, fix a Lie algebra L and let $I \trianglelefteq L$ and $K \leq L$. It is interesting to explore the relationship between the solvability of L, I and K.

Proposition 1.3.13 (Solvability Conditions).

1. *If L is solvable, then so is L/I.*
2. *If L is solvable, then so is K.*
3. *If I and L/I are solvable, then so is L.*

Proof. Let $\phi : L \twoheadrightarrow L/I$ be the quotient homomorphism.

1. Observe that it suffices to show that $\phi\left(L^{(i)}\right) = \phi(L)^{(i)}$ for all $i \in \mathbb{N}$: if this were true, then the existence of some $n \in \mathbb{N}$ such that $L^{(n)} = 0$ would imply that

$$\left(L/I\right)^{(n)} = \phi(L)^{(n)} = \phi\left(L^{(n)}\right) = \phi(0) = 0$$

making L/I solvable whenever L is.

We will now prove that $\phi\left(L^{(i)}\right) = \phi(L)^{(i)}$ by induction on i. When $i = 0$, the result is trivial: it is true that $\phi(L) = \phi(L)$ by reflexivity. Now, fix $i \in \mathbb{N}$ and assume that $\phi\left(L^{(i)}\right) = \phi(L)^{(i)}$.

Then,

$$\begin{aligned}\phi\left(L^{(i+1)}\right) = \phi\left(\left[L^{(i)}, L^{(i)}\right]\right) &= \phi\left(\mathrm{Span}\left(\left\{[x,y] \,\middle|\, x,y \in L^{(i)}\right\}\right)\right) \\ &= \mathrm{Span}\left(\phi\left(\left\{[x,y] \,\middle|\, x,y \in L^{(i)}\right\}\right)\right) \\ &= \mathrm{Span}\left(\left\{[\phi(x), \phi(y)] \,\middle|\, x,y \in L^{(i)}\right\}\right) \\ &= \left[\phi\left(L^{(i)}\right), \phi\left(L^{(i)}\right)\right] \\ &= \left[\phi(L)^{(i)}, \phi(L)^{(i)}\right] = \phi(L)^{(i+1)}\end{aligned}$$

as required.

2. It suffices to prove that for all $i \in \mathbb{N}$, $K^{(i)} \subseteq L^{(i)}$: if this were true, then the existence of some $n \in \mathbb{N}$ such that $L^{(n)} = 0$ would imply that $K^{(n)} = 0$, making K solvable whenever L is.

   We will now prove that $K^{(i)} \subseteq L^{(i)}$ by induction on i. The base case is trivial, because $K^{(0)} = K \subseteq L = L^{(0)}$. Now, fix $i \in \mathbb{N}$ and assume that $K^{(i)} \subseteq L^{(i)}$. Then,

$$\begin{aligned}K^{(i+1)} = \left[K^{(i)}, K^{(i)}\right] &= \mathrm{Span}\left(\left\{[x,y] \,\middle|\, x,y \in K^{(i)}\right\}\right) \\ &\subseteq \mathrm{Span}\left(\left\{[x,y] \,\middle|\, x,y \in L^{(i)}\right\}\right) \\ &= \left[L^{(i)}, L^{(i)}\right] = L^{(i+1)}\end{aligned}$$

   as required.

3. Let $m \in \mathbb{N}$ be such that $I^{(m)} = 0$ and let $n \in \mathbb{N}$ be such that $\left(L/I\right)^{(n)} = 0$. It suffices to prove that for all $i, j \in \mathbb{N}$, $\left(L^{(i)}\right)^{(j)} = L^{(i+j)}$: if this were true, then the fact that

$$\phi\left(L^{(n)}\right) = \left(L/I\right)^{(n)} = 0$$

   would immediately imply that $L^{(n)} \subseteq \ker(\phi) = I$, from which it would follow that $\left(L^{(n)}\right)^{(m)} = 0$, and therefore, that $L^{(n+m)} = 0$, making L solvable whenever I and L/I are.

   We will now prove that $\left(L^{(i)}\right)^{(j)} = L^{(i+j)}$ by letting i be arbitrary and performing induction on j. The base case is trivial, because $\left(L^{(i)}\right)^{(0)} = L^{(i)}$. Now, fix $j \in \mathbb{N}$ and assume that

$\left(L^{(i)}\right)^{(j)} = L^{(i+j)}$. Then,

$$\left(L^{(i)}\right)^{(j+1)} = \left[\left(L^{(i)}\right)^{(j)}, \left(L^{(i)}\right)^{(j)}\right] = \left[L^{(i+j)}, L^{(i+j)}\right] = L^{(i+j+1)}$$

as required.

□

We have similar results for nilpotency.

Definition 1.3.14 (Nilpotency of Subalgebras). We say a subalgebra of L is **nilpotent** if it is solvable as a Lie algebra in its own right.

As before, fix a Lie algebra L and let $I \trianglelefteq L$ and $K \leq L$. Imposing nilpotency conditions on L allows us to infer the same about L/I and K.

Proposition 1.3.15 (Nilpotency Conditions).

1. *If L is nilpotent, then so is L/I.*
2. *If L is nilpotent, then so is K.*

We will not prove these results here, as they are very similar to the corresponding results for solvability. We will, however, mention that the reason why we do not have a nilpotency condition for L when I and L/I are nilpotent is that it is not, in general, true that $\left(L^i\right)^j = L^{i+j}$ for $i, j \in \mathbb{N}$. The reason why this holds in the derived series is that the derived series is a recursive definition that *does not involve the base case*, meaning that $\left(L^{(i)}\right)^{(j)} = L^{(i+j)}$—that is, "taking the derived subalgebra i times and then taking it j times is tantamount to taking it $i+j$ times"—is simply a consequence of "doing a thing i times and then doing the same thing j times is tantamount to doing it $i+j$ times". In the case of the lower central series, however, the fact that the recursive definition *involves the base case* makes things problematic, because when we compute the jth lower central ideal of L^i, *we have a different base case*: we are computing Lie brackets with respect to L^i instead of L, meaning that we are "doing a thing i times and then doing a *different* (if analogous) thing j times". We see this clearly in the following counterexample.

Counterexample 1.3.16 ($\mathfrak{r}_2$, again). $\mathfrak{r}_2$ has an ideal I of dimension and co-dimension 1 spanned by the matrix E_{12} with a 1 in the 12 entry and 0s everywhere else (we have already indirectly shown this in Example 1.3.4, so we do not do so explicitly here). Since the dimension and co-dimension of I are 1, both I and $\mathfrak{r}_2/_I$ are abelian, making them nilpotent. However, $\mathfrak{r}_2$ is not nilpotent, as we have already shown in Counterexample 1.3.10.

As one might expect, this is connected to the above discussion. For all $i, j \in \mathbb{N}$, if $i \geq 1$, then $\mathfrak{r}_2^{i+j} = I$. However, $\mathfrak{r}_2^i = I$ as well, and I is abelian, meaning that $I^j = 0$ if additionally $j \geq 1$. Therefore, for all $i, j \geq 1$, we have $\left(\mathfrak{r}_2^i\right)^j = 0 \neq I = \mathfrak{r}_2^{i+j}$.

It turns out that solvability also tells us about the derived subalgebra and codimesions. We begin with a simple observation.

Lemma 1.3.17. *$L = L'$ if and only if $\forall i \in \mathbb{N}$, $L^{(i)} = L^{(1)} = L' = L$.*

Proof. One direction is trivial, so we do not bother to prove it. To prove that if L equals its derived subalgebra then L equals all subsequent derived subalgebras, we argue by induction on i. The base case is trivial, because $L^{(0)} = L$. Now, fix $i \in \mathbb{N}$ and assume that $L^{(i)} = L$. Then,

$$L^{(i+1)} = \left[L^{(i)}, L^{(i)}\right] = [L, L] = L'$$

Furthermore, it is clear that $L^{(1)} = L'$, and, by assumption, $L' = L$. This completes the induction and proves the desired result for all $i \in \mathbb{N}$. □

There is an immediate (and somewhat trivial) consequence.

Corollary 1.3.18. *If $L \neq 0$, then L is solvable if and only if $L' < L$.*

Proof. We know, from Lemma 1.3.17, that $L' = L$ if and only if $L^{(i)} = L$ for all $i \in \mathbb{N}$. In particular, since L is nonzero, none of the $L^{(i)}$ can be zero, which is true if and only if L is not solvable. Therefore, L is solvable if and only if $L' \neq L$, which is equivalent to $L' < L$. □

There is also a less immediate consequence that comes from applying a combination of Corollary 1.3.18 and the Correspondence Theorem (Theorem 1.1.36) to the ideals of quotient spaces of solvable Lie algebras.

Corollary 1.3.19. *If $L \neq 0$ and L is solvable, there exists an ideal $I \trianglelefteq L$ of codimension 1.*

Proof. Consider the quotient Lie algebra $K := {}^{L}/_{L'}$. We know that $0 < K$, because $K = 0$ would imply that $L = L'$, which is impossible because L is solvable, as shown in Corollary 1.3.18. Therefore, K contains a subspace W of dimension 1. Since K is abelian, Proposition 1.1.27 tells us that W is an ideal of K. The Correspondence Theorem (Theorem 1.1.36) then tells us that the preimage V of W under the quotient epimorphism is an ideal of L that contains L'. Simple arithmetic and dimension results from linear algebra then tell us

$$\dim(V) = \dim(W) + \dim(L') = (\dim(L) - \dim(L') - 1) + \dim(L') = \dim(L) - 1$$

□

1.3.3 The Radical Ideal

Throughout this subsection, we will assume that L is finite-dimensional.

We begin with a basic result about the sums of ideals.

Lemma 1.3.20. *Let $I, J \trianglelefteq L$. For all $k \in \mathbb{N}$, we have*

$$(I + J)^{(k)} \subseteq I^{(k)} + J^{(k)}$$

Proof. We argue by induction on k. The base case is trivial, because $(I+J)^{(0)} = I+J = I^{(0)}+J^{(0)}$. Now, fix $k \in \mathbb{N}$ and assume that $(I + J)^{(k)} \subseteq I^{(k)} + J^{(k)}$. Then,

$$\begin{aligned}
(I + J)^{(k+1)} &= \left[(I + J)^{(k)}, (I + J)^{(k)}\right] \\
&\subseteq \left[I^{(k)} + J^{(k)}, I^{(k)} + J^{(k)}\right] \\
&\subseteq \left[I^{(k)}, I^{(k)}\right] + \left[I^{(k)}, J^{(k)}\right] + \left[J^{(k)}, I^{(k)}\right] + \left[J^{(k)}, J^{(k)}\right] \\
&\subseteq I^{(k+1)} + J^{(k+1)}
\end{aligned}$$

as required. □

Lemma 1.3.21. *Let $I, J \trianglelefteq L$. If I and J are solvable, then so is $I + J \trianglelefteq L$.*

Proof. Let $n \in \mathbb{N}$ be such that $I^{(n)} = 0$ and let $m \in \mathbb{N}$ be such that $J^{(m)} = 0$. Since $m + n \geq m, n$ and $I^{(n)} = J^{(m)} = 0$, we know that $I^{(n+m)} = J^{(n+m)} = 0$. Then, applying Lemma 1.3.20, we have

$$(I + J)^{(n+m)} \subseteq I^{(n+m)} + J^{(n+m)} = 0 + 0 = 0$$

proving that the derived series of $I + J$ eventually stabilises at 0. Therefore, $I + J$ is solvable. □

Corollary 1.3.22. *There exists a solvable ideal of L that contains all other solvable ideals of L.*

Proof. Let R be a solvable ideal of L of maximal dimension.[2] Now, fix any $I \trianglelefteq L$. Lemma 1.3.21 tells us that $I + R$ is a solvable ideal of L. But, since R is of maximal dimension, we know that $\dim(I + R) \leq \dim(R)$. Therefore, we must have that $I + R = R$. Then, we must have that $I \subseteq R$, as any $i \in I$ is expressible as $i + 0$, and since $0 \in R$, we have $i = i + 0 \in I + R = R$. Hence, $I \subseteq R$, proving that R contains any ideal of L that is solvable. □

This solvable ideal has a name.

Definition 1.3.23 (Radical Ideal). The **radical ideal** of L is the solvable ideal of L that contains all other solvable ideals of L, which we know exists from Corollary 1.3.22.

Given that we frame simplicity in terms of normal subgroups in group theory, and given that their analogue in the theory of Lie algebras is ideal, we will define semi-simplicity to be a weak form of simplicity, where the non-trivial ideals that are not allowed to exist are *semi-simple*.

Definition 1.3.24 (Semi-Simplicity). We say that L is **semi-simple** if its radical ideal is the 0 ideal.

[2]When we say maximal dimension, we mean that the dimension of R is the largest possible dimension such that a solvable ideal of that dimension exists. This is well-defined because L is finite-dimensional, and the dimension of any ideal of L is necessarily $\leq \dim(L)$.

It is clear, from the definition of the radical ideal, that semi-simple Lie algebras are precisely those that contain no nontrivial solvable ideals (and are not solvable themselves).

Our aim for this module will be to classify all semi-simple Lie algebras over $\mathbb{C}$. We will do this by first classifying all solvable Lie algebras and then using that classification to classify all semi-simple Lie algebras. We will need a *lot* more machinery before we can do this, but we will get there eventually.

While we have already alluded to the following definition, we will mention it explicitly for completeness. It is no different from what we would expect in groups.

Definition 1.3.25 (Simplicity). We say that L is **simple** if it has no nontrivial ideals.

As we have mentioned, semi-simple Lie algebras contain no nontrivial *solvable* ideals (and are not solvable themselves). Simple Lie algebras contain no nontrivial ideals of *any* kind, solvable or otherwise. Hence, semi-simple Lie algebras are simple, and semi-simplicity is a strictly weaker notion than simplicity.[3]

1.3.4 Ascending Series of Ideals

We will end by talking about ascending series of ideals and their corresponding quotients. Throughout this subsection, L will denote an arbitrary Lie algebra.

Definition 1.3.26 (Ascending Central Series). We say that an increasing chain of ideals

$$0 \subseteq L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$$

is an **ascending central series** of L if $L_1 = Z(L)$ and for all $i \in \mathbb{N}$, we have

1. $L_i \trianglelefteq L$ with quotient map $g_i : L \twoheadrightarrow {}^{L}/_{L_i}$
2. $L_{i+1} = g_i^{-1}(Z({}^{L}/_{L_i}))$

[3]Some authors require that a simple Lie algebra have dimension ≥ 1, thereby excluding the 0 Lie algebra from being simple. Under this definition, 0 would be semi-simple but not simple, unless we also required the dimension to be at least 1 in the definition of semi-simplicity. These are minutia, which would be very important if we were to study Lie algebras in a very formal setting, but there is no need to consider these cases in this module.

Convention. We will use subscripted L_is to denote elements of the ascending central series, in contrast to superscripts used for the descending central series.

We now have an equivalent criterion for nilpotency.

Proposition 1.3.27. *L is nilpotent if and only if $L_n = L$ for some $n \in \mathbb{N}$.*

Proof.

($\Longrightarrow$) One can show by induction on n that if $L^n = 0$, then $L_n = L$. Then, if $\exists n \in \mathbb{N}$ such that $L^n = 0$, ie, if L is nilpotent, then $L_n = 0$ as well. `sorry`

($\Longleftarrow$) `sorry`

□

We end with another counterexample that shows that solvable Lie algebras need not be nilpotent.

Counterexample 1.3.28. For all n, $\mathfrak{t}(n)$ is solvable but not nilpotent.

Proof that $\mathfrak{t}(n)$ is solvable. First, observe that $[\mathfrak{t}(n), \mathfrak{t}(n)] = \mathfrak{t}(n)' = \mathfrak{u}(n)$. By `sorry`, we know that $\mathfrak{u}(n)$ is nilpotent. Therefore, $\mathfrak{u}(n)$ is solvable. Furthermore, $\mathfrak{t}(n)/\mathfrak{t}(n)'$ is abelian, making it solvable by `sorry`. Therefore, by Proposition 1.3.13, $\mathfrak{t}(n)$ is solvable. □

Proof that $\mathfrak{t}(n)$ is not nilpotent. □

1.4 Subalgebras of $\mathfrak{gl}(n)$

We now turn our attention to the structure of subalgebras of $\mathfrak{gl}(n)$ for some fixed $n \in \mathbb{N}$. We will begin by developing some mroe general theory, following which we will prove important theorems about the structure of such subalgebras.

1.4.1 Induced Actions on Quotients by Invariant Subspaces

We will begin by recalling the linear algebraic theory of invariant subspaces and adapt that theory to the context of Lie algebras. Throughout this subsection, we will fix a subset $L \subseteq \mathfrak{gl}(n)$ and a subspace $U \leq \mathbb{C}^n$.

Definition 1.4.1 (Invariance)**.** We say that U is **L-invariant** if for all $T \in L$ and $v \in U$, we have $T(v) \in U$.

The action of L on $\mathbb{C}^n$ induces a natural action on the quotient space of $\mathbb{C}^n$ by U.

Definition 1.4.2 (Induced Action)**.** To each $T \in L$, we can associate the linear map $\overline{T} : \mathbb{C}^n/U \to \mathbb{C}^n/U$ defined by

$$\overline{T}(v + U) = T(v) + U \tag{1.4.1}$$

for all $v + U \in \mathbb{C}^n/U$. We will refer to the map $T \mapsto \overline{T}$ as the **induced action** of L on $\mathbb{C}^n/U$.

Indeed, when L is a Lie subalgebra of $\mathfrak{gl}(n)$, we can go one step further.

Proposition 1.4.3. *If L is a Lie subalgebra of $\mathfrak{gl}(n)$, the induced action map $\Phi : L \to \mathfrak{gl}(\mathbb{C}^n/U) : T \mapsto \overline{T}$ is a Lie algebra homomorphism.*

Proof. sorry □

1.4.2 Linear Algebraic and Lie Algebraic Nilpotency

Recall the following definition from linear algebra.

Definition 1.4.4 (Nilpotency of Elements)**.** We say that $x \in \mathfrak{gl}(n)$ is **nilpotent** if there exists an $m \in \mathbb{N}$ such that $x^m = 0$.

We can extend this to sub-vector spaces.

Definition 1.4.5 (Nilpotency of Subspaces)**.** We say a sub-vector space $N \leq \mathfrak{gl}(n)$ is **nilpotent** if every element of N is nilpotent.

We can say something about the adjoint of a nilpotent element.

Lemma 1.4.6. *Let $x \in \mathfrak{gl}(n)$ be nilpotent. Then, $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{gl}(n))$ is nilpotent as well.*

Proof. We need to show that there exists an $m \in \mathbb{N}$ such that the map we get by successively composing the adjoint map $\mathrm{ad}(x)$ m times is identically zero.

Fix $y \in \mathfrak{gl}(n)$. Then,

$$\begin{aligned} \mathrm{ad}(x)(y) &= [x, y] = xy - yx \\ \mathrm{ad}(x)^2(y) &= [x, [x, y]] = x\,[x, y] - [x, y]\,x = x^2y - xyx - xyx + yx^2 \\ \mathrm{ad}(x)^3(y) &= [x, [x, [x, y]]] = x^3y + \cdots + xyx^2 - yx^3 \end{aligned}$$

More generally, one can show that

$$\mathrm{ad}(x)^m(y) = \sum_{i=0}^{m} \lambda_{i,m} x^i y x^{m-i}$$

for all $m \in \mathbb{N}$ and some $\lambda_{i,m} \in \mathbb{Z}$. In particular, since all powers of x beyond some m are zero, we have that $\mathrm{ad}(x)^m(y) = 0$ for all $y \in \mathfrak{gl}(n)$. □

We have an important relationship between linear algebraic and lie algebraic nilpotency of a Lie subalgebra.

Theorem 1.4.7 (Engel's Theorem)**.** *Let N be a Lie subalgebra of $\mathfrak{gl}(n)$. If N is nilpotent as a sub-vector space of $\mathfrak{gl}(n)$, then there exists a basis of $\mathbb{C}^n$ with respect to which every element of N is upper-triangular.*

Before proving Engel's Theorem, we will state and prove the following Corollary that underscores the significance of this result.

Corollary 1.4.8. *Any nilpotent sub-vector space of $\mathfrak{gl}(n)$ is also nilpotent as a Lie subalgebra.*

Proof. Let N be a nilpotent sub-vector space of $\mathfrak{gl}(n)$. By Engel's Theorem, there exists a basis of $\mathbb{C}^n$ with respect to which every element of N is upper-triangular. In particular, they must all have zeros on the diagonal, because they are nilpotent: they are of the form

$$\begin{bmatrix} 0 & & * \\ \vdots & \ddots & \\ 0 & \cdots & 0 \end{bmatrix}$$

sorry □

For the remainder of this subsection, we will focus on proving Engel's Theorem. We will fix a nilpotent subspace $N \leq \mathfrak{gl}(n)$. The high-level idea is to perform induction on $\dim(L)$ and draw a parallel with the proof of the Jordan Canonical Form theorem[4]. We will first show that it suffices to show that a certain distinguished vector exists, following which we will show that it does.

For the remainder of this subsection, we will denote the **simultaneous kernel** of all elements of N by

$$U_n := \{v \in \mathbb{C}^n \mid \forall T \in N,\ T(v) = 0\} = \bigcap_{T \in N} \ker(T) \tag{1.4.2}$$

As an intersection of sub-vector spaces, U_n is a subspace of $\mathbb{C}^n$. Furthermore, U_n (and, by extension, all of its subspaces) are N-invariant: for all $T \in N$ and $v \in U_n$, we have $T(v) = 0 \in U_n$.

We are now ready to reduce the proof of Engel's Theorem to showing that all the elements of T have a common eigenvector with eigenvalue 0—or, equivalently, to showing that U_n is nonzero.

Lemma 1.4.9. *If U_n contains a nonzero element, then there exists a basis of $\mathbb{C}^n$ with respect to which every element of N is upper-triangular.*

[4] Remember, we are working over $\mathbb{C}$.

Proof. We argue by induction on n. When $n = 1$, the result is trivial: every element of N (and of $\mathfrak{gl}(n) = \mathfrak{gl}(1)$) is upper-triangular, so the fact that $U_n = 0$ is not a problem. Now, suppose that there exists a nonzero element $v \in U_n$, ie, such that $T(v) = 0$ for all $T \in N$. Let $V = \mathbb{C}^n / \text{Span}(v)$. Since $\text{Span}(v) \leq U_n$ and U_n is N-invariant, we know that $\text{Span}(v)$ is N-invariant as well, allowing us to develop the machinery developed in Section 1.4.1 that tells us about the Lie algebraic properties of $\mathfrak{gl}(V)$.

From Proposition 1.4.3, we know that the map that takes $T \in N$ to its induced action on V is a Lie algebra homomorphism. Therefore, by Lemma 1.1.12, its image is a subalgebra of $\mathfrak{gl}(V)$. Indeed, the elements of this subalgebra consists of nilpotent elements: for any $T \in \mathfrak{gl}(V)$, we know there exists some $m \in \mathbb{N}$ such that $T^m = 0$, and the same m will work for the induced action $\overline{T}$ of T on the quotient space: for any $x + \text{Span}(v) \in \mathbb{C}^n / \text{Span}(v)$,

$$\overline{T}^m(x + U) = \overline{T}^m(x) + U = 0 + U$$

Therefore, by the induction hypothesis, there exists a basis

$$\overline{B} := \{v_1 + \text{Span}(v), \ldots, v_{n-1} + \text{Span}(v)\}$$

of V with respect to which every element of the image of N is upper-triangular. We can then lift this basis to a basis and this basis

$$B := \{v_1, \ldots, v_{n-1}, v\}$$

of $\mathbb{C}^n$ by adding v to it. B has the desired property that every element of N is upper-triangular with respect to it. □

The way we will prove Engel's Theorem is to construct a sequence of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m = \mathbb{C}^n$$

such that $N(V_i) \subseteq V_{i-1}$. We will refine this sequence so that $m = n$, ie, so that

$$\dim\left(V_i / V_{i-1}\right) = 1$$

using Corollary 1.3.19. We can then take distinguished elements from each of the quotients to form a basis of $\mathbb{C}^n$, and this will be the basis with respect to which every element of N is upper-triangular.

We will now show that U_n is, indeed, nonzero.

Lemma 1.4.10. *There exists a nonzero vector $v \in U_n$.*

Proof. We need to show that $T(v) = 0$ for all $T \in N$. We argue by induction on $\dim(N)$. The base case $\dim(N) = 1$ is clear: N must be the span of a single, nilpotent element, which necessarily has a (nonzero) eigenvector with eigenvalue 0. So, assume N is such that for all Lie subalgebras of $\mathfrak{gl}(n)$ of dimension less than $\dim(N)$, there exists a nonzero vector in the simultaneous kernel U_n of all elements of N.

Let $A \subseteq N$ be a maximal[5], proper Lie subalgebra of N. Consider the map $\phi : A \to \mathfrak{gl}({}^{N}/_{A})$ that maps any $g \in A$ to the map that sends every $T + A \in {}^{N}/_{A}$ to the map $[g, T] + A \in {}^{N}/_{A}$, where the map $[g, T]$ is the map $gT - Tg$.

Observe that since $\dim(\phi(A)) \leq \dim(A)$ and $\dim(A) < \dim(L)$ by the assumption that A is proper, we know that $\dim(A) < L$. Therefore,we can apply the induction hypothesis to A. `sorry` □

There is also a more general formulation of Engel's Theorem over arbitrary Lie algebras.

Theorem 1.4.11 (Engel's Theorem, Second Version)**.** *Let L be an arbitrary Lie algebra. Then, L is nilpotent if and only if for all $x \in L$, the adjoint map $\mathrm{ad}(x)$ is nilpotent.*

Proof. Assume that L is nilpotent. Then, there exists some $m \in \mathbb{N}$ such that $L^m = 0$. Then, any composition of Lie brackets of length m is zero: for all $x, y \in L$,

$$\mathrm{ad}(x)^n(y) = [x, [x, \cdots [x, y] \cdots]] = 0$$

This gives us one direction of the proof.

For the converse direction, we apply Theorem 1.4.7 (the standard formulation of Engel's Theorem) to the `sorry` □

[5] with respect to inclusion

1.4.3 Weights of Lie Algebras

Throughout this subsection, let V be a finite-dimensional $\mathbb{C}$-vector space, L a lie subalgebra of $\mathfrak{gl}(V)$, and $\lambda : L \to \mathbb{C}$ be an arbitrary function.

Definition 1.4.12 (Weight Space). We say the **weight space** of λ with respect to L is the space

$$V_\lambda := \{v \in V \mid \forall T \in L,\ T(v) = \lambda(T) \cdot v\}$$

The weight space gives us useful information about L and λ.

Lemma 1.4.13. *If V_λ is nonzero, then λ is a linear map.*

Proof. Suppose that $V_\lambda \neq 0$. Fix $S, T \in L$. We know there exists a nonzero vector $v \in V$ such that $S(v) = \lambda(S) \cdot v$ and $T(v) = \lambda(T) \cdot v$. In particular, we have that

$$\lambda(S+T) \cdot v = (S+T)(v) = S(v) + T(v) = \lambda(S) \cdot v + \lambda(T) \cdot v = (\lambda(S) + \lambda(T)) \cdot v$$

Given that $\lambda(S+T)$ and $\lambda(S) + \lambda(T)$ are both scalars, we must have that $\lambda(S+T) = \lambda(S) + \lambda(T)$. □

Lemma 1.4.14. *V_λ is a sub-vector space of V.*

Proof. sorry □

Weight spaces are rather interesting, and in the remainder of this subsection, we will prove a lemma that will help us prove the very important Theorem 1.4.21, which we shall see shortly.

Lemma 1.4.15 (The Invariance Lemma). *Let A be an ideal of L and let $\lambda : A \to \mathbb{C}$ be a weight on A. Then, the weight space*

$$V_\lambda = \{v \in V \mid \forall a \in A,\ a(v) = \lambda(a) \cdot v\}$$

of I is an L-invariant subspace of V.

Proof. Fix $y \in L$ and $w \in V_\lambda$. We want to show that $y(w) \in V_\lambda$, ie, that $y(w)$ is an eigenvector of every $a \in A$, with corresponding eigenvalue $\lambda(a)$. □

We end by computing all the weights and weight spaces of a few subalgebras of $\mathfrak{gl}(n)$.

Example 1.4.16. Let I denote the $n \times n$ identity matrix. Consider the set

$$\{\lambda \cdot I \mid \lambda \in \mathbb{C}\}$$

One can show that this is a subalgebra of $\mathfrak{gl}(n)$. In this case, there is only one weight: this is the map that sends every element $\lambda \cdot I$ of the above subalgebra to the corresponding λ. The weight space of this weight is all of $\mathbb{C}^n$.

Example 1.4.17. Let $n = 3$, and let E_{ij} denote the 3×3 matrix whose only nonzero entry is a 1 in the ith row and jth column. Consider the set

$$\left\{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C} \right\}$$

One can show that the above set is a subalgebra.

The key to computing the weights and weight spaces is to consider the eigenvectors of the elements of the subalgebra. Clearly, a basis of the above is $\{E_{11} + E_{22}, E_{33}\}$. The two weights are the maps that send each of these to 1. sorry

We can apply the above to compute the weights of the upper-triangular Lie subalgebra.

Example 1.4.18 ($\mathfrak{t}(n)$)**.** Consider the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$. We know that $\mathrm{Span}(e_1)$ is an eigenvector of every element of $\mathfrak{t}(n)$. Indeed, one can show (sorry) that it is the *only* such simultaneous eigenvector. Therefore, the only weight is the one that maps any element of $\mathfrak{t}(n)$ to the element that lives in its first row and first column, which is precisely the eigenvalue of e_1 under its action. sorry

Finally, we underscore the importance of weight spaces by mentioning that they can be used to prove a seemingly unrelated fact about matrices.

Lemma 1.4.19. *Let $A, B \in \mathfrak{gl}(n)$ be diagonalisable. If A and B commute, then there exists a basis with respect to which both A and B are diagonalisable.*

Proof. Consider the subspace $L := \mathrm{Span}(A, B) \leq \mathfrak{gl}(n)$. Observe that since A and B commute, $[A, B] = 0$. Therefore, L is an abelian Lie subalgebra of $\mathfrak{gl}(n)$. The idea is that the weight space of □

Corollary 1.4.20. *Let $A, B \in \mathfrak{gl}(n)$ be diagonalisable. If A and B commute, then $A + B$ is diagonalisable.*

Proof. Rewrite A and B in the basis given in the previous lemma. Their sum is then a sum of diagonal matrices, which is diagonal (in that same basis). □

1.4.4 Lie's Theorem

In this subsection, we discuss a result similar to Engel's Theorem, but for *solvable* Lie algebras instead of nilpotent ones. Throughout, we fix a **solvable** subalgebra $L \leq \mathfrak{gl}(n)$ for some $n \in \mathbb{N}$.

Theorem 1.4.21 (Lie's Theorem). *There exists a basis of $\mathbb{C}^n$ such that every element of L is upper-triangular with respect to it.*

Before proceeding with the proof of Lie's Theorem, we will prove a corollary that underscores the significance of this result.

Corollary 1.4.22. *L' is solvable.*

Proof. Consider the adjoint map $\mathrm{ad} : L \to \mathfrak{gl}(L)$. We know that $\mathrm{ad}(L)$ is solvable by sorry; therefore, by Lie's Theorem, it is contained in $\mathfrak{t}(L)$. sorry □

Our proof strategy will be to obtain some $\lambda \in L^*$, ie, a linear function $L \to \mathbb{C}$, such that ${\mathbb{C}^n}_\lambda$. We will repeat a simpler version of the proof of Engel's Theorem: we will perform induction on n, applying the induction hypothesis to the image of L under the quotient epimorphism $\Phi_\lambda : L \twoheadrightarrow \mathfrak{gl}\left(\mathbb{C}^n / \mathbb{C}^n_\lambda\right)$. $\Phi_\lambda(L)$ is solvable, because it is a quotient of a solvable Lie algebra. We will then be able to apply the fact that

$$\dim(\Phi_\lambda(L)) \leq \dim(\mathbb{C}^n) - \dim(\mathbb{C}^n_\lambda) < \dim(\mathbb{C}^n) = n$$

because $\mathbb{C}^n_\lambda \neq 0$, allowing us to apply the induction hypothesis on $\Phi_\lambda(L)$.

sorry

We end with an example that shows that Lie's Theorem is not necessarily true over fields of prime characteristic.

Counterexample 1.4.23 (Lie's Theorem Fails over Prime Characteristic). Let p be a prime number and let F be a field of characteristic p. Consider the vector space F^p, and let $\{e_1, \ldots, e_p\}$ denote a basis of it. We know that $\mathfrak{gl}(F^p)$, the set of F-linear maps from F^p to itself, is a Lie algebra over F with the commutator bracket. Denote it by L for the purposes of this example.

Consider the following elements of L:

$$x := e_i \mapsto i \cdot e_i$$
$$y := \begin{cases} e_i \mapsto e_{i+1} & \text{if } i < p \\ e_p \mapsto e_1 \end{cases} \in L$$

We will show that x and y have no common eigenvectors, a fact we can use to generate a basis

This is more of an aside, since we are primarily interested in complex Lie algebras in this module. Nevertheless, we mention it here because it is interesting.

Chapter 2

Representations of Lie Algebras

Throughout this chapter, L will denote a finite-dimensional Lie algebra over $\mathbb{C}$, unless we state otherwise.

2.1 Important Definitions and First Examples

2.1.1 Representations and Modules

Definition 2.1.1 (Representation). A **representation** of L is a Lie algebra homomorpshism $\rho : L \to \mathfrak{gl}(V)$ for some finite-dimensional $\mathbb{C}$-vector space V.

Example 2.1.2 (The Adjoint Representation). The adjoint map $\text{ad} : L \to \mathfrak{gl}(L)$ is a representation: Proposition 1.1.39 tells us it is a Lie algebra homomorphism.

If $\rho : L \to \mathfrak{gl}(V)$ is a representation, then we can define a map $(\ell, v) \mapsto \rho(\ell)(v) : L \times V \to V$. We can use this to define a Lie algebra module, similar to the concept of group modules when defining complex representations thereof.

Definition 2.1.3 (Lie Module). A **Lie module**, or L-**module**, is a finite-dimensional vector space V with a pairing $\rho : L \times V \to V$ such that

1. ρ is $\mathbb{C}$-bilinear.

2. $\rho([a, b], v) = \rho(a, \rho(b, v)) - \rho(b, \rho(a, v))$.

Indeed, one can show that any representation $\rho : L \to V$ satisfies the above properties with respect to the pairing $(\ell, v) \mapsto \rho(\ell)(v)$ and that any Lie algebra module V with pairing ρ admits a uniquely defined representation $\ell \mapsto \rho(\ell, \cdot) : L \to \mathfrak{gl}(V)$. Thus, the two concepts are equivalent.

Convention. We will abuse notation and not distinguish the notions of representations and modules. Similarly, we will not split hairs about the notation for the two: $\rho(\cdot, \cdot)$ should be interpreted as meaning the same thing as $\rho(\cdot)(\cdot)$.

Indeed, as with representations of groups and group modules, we have an equivalence of categories between the category of representations of L and that of L-modules. One would need to argue a bit more rigorously, by defining the morphisms in each one, but we will not do this and take this as an implicit fact.

2.1.2 Homomorphisms, Submodules and Quotient Modules

Throughout this subsection, we denote by M an L-module of finite $\mathbb{C}$-dimension.

Definition 2.1.4 (Lie Submodule). An L-**submodule** of M is a sub-vector space $N \leq M$ that is invariant under the action of L, ie,

$$\forall l \in L, x \in N, \ \ l \cdot x \in N$$

`sorry`

2.1.3 Simple or Irreducibile L-Modules

Throughout this subsection, we denote by M an L-module of finite $\mathbb{C}$-dimension.

Definition 2.1.5 (Irreducibility). M is **irreducible** if it has no proper, non-zero submodules.

Remark. We can define a similar notion of irreducibility of representations, which we can show to be equivalent to the irreducibility of the corresponding module. This is why we choose to call a module that is **simple**, ie, that satisfies Definition 2.1.5, 'irreducible'.

Nevertheless, we use the following convention.

Convention. We will use the terms 'simple' and 'irreducible' interchangeably.

There exist a large number of irreducible modules of any Lie algebra.

Example 2.1.6. Any one-dimensional L-module is irreducible.

We remind the reader of the famed Jordan-Hölder Theorem from commutative algebra. We will mention that all Lie algebra modules admit composition series. We will combine this existence result with the standard formulation of the Jordan-Hölder Theorem into the following theorem.

Theorem 2.1.7 (Jordan-Hölder). *If $M \neq 0$, then there exists a sequence*

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M \tag{2.1.1}$$

of submodules of M such that M_i / M_{i-1} is irreducible for all i. Furthermore, each M_i / M_{i-1} is unique up to permutation and isomorphism—that is, they do not depend on the sequence itself, but only on the isomorphism class of M.

Finally, we mention some useful results on irreducible representations of nilpotent and solvable Lie algebras that follow from Engel's Theorema and Lie's Theorem respectively.

Proposition 2.1.8. *Let L be solvbale. Then, every irreducible L-module is one-dimensional over $\mathbb{C}$.*

Proof. `sorry` □

2.1.4 Semi-Simple L-Modules

We have yet more parallels with the representation theory of finite groups.

Definition 2.1.9 (Direct Sum). The **direct sum** of two L-modules M and N is the L-

module $M \oplus N$ with the action defined by

$$\rho(\ell)(m, n) = (\rho(\ell)(m), \rho(\ell)(n)) \tag{2.1.2}$$

We then have a notion of semi-simplicity for L-modules, which is essentially the same as that of any module.

Definition 2.1.10 (Semi-Simplicity). An L-module M is **semi-simple** if it is a direct sum of simple (irreducible) L-modules.

The following theorem, which is quite hard to prove, explains the reason we define semi-simplicity of Lie algberas the way we do in Definition 1.3.24.

Theorem 2.1.11. *If L is semi-simple, then every finite-dimensional L-module is semi-simple.*

We do not prove this theorem here, but we will take it for granted going forward.

Example 2.1.12. Let L be solvable. Theorem 1.4.21 (Lie's Theorem) tells us that `sorry`

2.2 The Adjoint Representation and the Killing Form

In this section, we will explore some of the properties of the adjoint representation, which is closely related to the killing form, a bilinear form on L that will help us better understand its structure.

2.2.1 Properties of the Adjoint Representation

We begin with a result on the Jordan decomposition of the adjoint representation.

Recall that the Jordan decomposition of a linear map T (from some $\mathbb{C}$-vector space to itself) involves expressing T as $d + n$, where d is diagonalisable, n is nilpotent and $dn = nd$. In particular, d and n are simultaneously triangularisable[1], meaning that there exists some basis, known as a

[1] That all commuting linear maps are simultaneously triangularisable is a well-known fact from linear algebra, and we do not prove it here.

Jordan basis of T, with respect to whichthe matrix of T is upper-triangular, with all of its diagonal entries being its eigenvalues (given by d) and its super-diagonal entries being either 0 or 1 in a nilpotent manner (given by n).

We can show that the adjoint representation of a Lie algebra respects this decomposition. To that end, we show that it respects diagonalisability, nilpotency and commutativity.

We begin with diagonalisability.

Lemma 2.2.1. *Let $d \in \mathfrak{gl}(n)$ be diagonal with respect to some basis. Then, $\operatorname{ad} d$ is diagonalisable.*

Proof. sorry □

Recall that Lemma 1.4.6 tells us precisely that the adjoint representation respects nilpotency. Finally, we can use the fact that the adjoint representation is a Lie algebra homomorphsim to show that it respects commutativity.

Lemma 2.2.2. *If $x, y \in \mathfrak{gl}(n)$ are such that $xy = yx$, then $\operatorname{ad}(x)\operatorname{ad}(y) = \operatorname{ad}(y)\operatorname{ad}(x)$.*

Proof. Observe that $xy = yx \iff [x, y] = 0$. By Proposition 1.1.39, we have that

$$[\operatorname{ad}(x), \operatorname{ad}(y)] = \operatorname{ad}([x, y])$$

Since ad is linear and $[x, y] = 0$, we have that $[\operatorname{ad}(x), \operatorname{ad}(y)] = 0$, or, equivalently, that $\operatorname{ad}(x)\operatorname{ad}(y) = \operatorname{ad}(y)\operatorname{ad}(x)$. □

Finally, we end with a discussion on how the adjoint representation looks in terms of commutative diagrams.

Proposition 2.2.3. *Let K be another Lie Algebra. Then,*

$$\begin{array}{ccc} L \oplus K & \xrightarrow{\mathrm{ad}_{L\oplus K}} & \mathfrak{gl}(L\oplus K) \\ & \searrow^{\mathrm{ad}_L \oplus \mathrm{ad}_K} & \uparrow \\ & & \mathfrak{gl}(L)\oplus\mathfrak{gl}(K) \end{array} \tag{2.2.1}$$

Proof. sorry □

In particular, (2.2.1) gives us the following relationship between the traces of the adjoints: sorry

2.2.2 The Killing Form

We are now ready to define the killing form on L.

Definition 2.2.4 (The Killing Form)**.** The **killing form** on L is the map $\kappa : L \times L \to \mathbb{C}$ defined by

$$\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}(x) \cdot \mathrm{ad}(y)) \tag{2.2.2}$$

where $\mathrm{ad} : L \to \mathfrak{gl}(L)$ denotes the adjoint representation of L.

Convention. For the remainder of this chapter, we will denote the killing form on L by κ.

The basic properties of the killing form come from the following.

Proposition 2.2.5. *κ is a symmetric, bilinear form on L.*

We will not prove this proposition, as it involves checking basic facts from linear algebra. We will take it for granted going forward.

We will now prove some identities about the killing form. We will begin by stating a basic identity involving the trace.

Lemma 2.2.6. *For all $A, B, C \in \mathfrak{gl}(L)$, we have that*

$$\mathrm{Tr}([A, B], C) = \mathrm{Tr}(A, [B, C]) \tag{2.2.3}$$

Proof. The proof is a simple consequence of two facts: first, that matrix multiplication is associative, and second, that the trace of a product of two matrices is invariant under swapping them. We will leave the details to the reader. □

This gives us a similar identity for the killing form.

Corollary 2.2.7. *For any $A, B, C \in L$, we have that*

$$\kappa([A, B], C) = \kappa(A, [B, C]) \tag{2.2.4}$$

Proof. `sorry` □

2.2.3 The Killing Form on Ideals and Subalgebras

Seeing as there is a Killing Form defined on any Lie algebra, and seeing as ideals and subalgebras are also Lie algebras in their own right, we can define Killing Forms on them as well. A natural question to ask is whether these are related to the Killing Form on the Lie algebra in which they live. We will show that this is indeed the case for ideals, but not necessarily for subalgebras.

For the remainder of this subsection, for any Lie algebra $\mathfrak{L}$, we will denote the Killing Form on it by $\kappa_{\mathfrak{L}}$.

Proposition 2.2.8. *Let $I \trianglelefteq L$. Then,*

1. *$\kappa_L|_{I \times I} = \kappa_I$, ie, the restriction of the Killing Form on L to inputs in I is equal to the Killing Form on I.*
2. *The orthogonal complement of I with respect to κ_L is also an ideal of L.*

We can show that the results of Proposition 2.2.8 fail for subalgebras that are not ideals.

Counterexample 2.2.9. Let $h = \text{diag}(-1, 1) \in \mathfrak{gl}(2)$, and let $H = \text{Span}(h)$ be the Lie subalgebra generated by h. We know that H is abelian, meaning that the Killing Form κ_H is identically zero. However, we can show that $\kappa_L(h, h) \neq 0$, proving that $\kappa_L|_{H \times H} \neq \kappa_H$.

Finally, we can say something about the Killing Form on the direct sum of two Lie Algebras.

Proposition 2.2.10. *The Killing Form of the direct sum of two Lie Algebras is the sum of their Killing Forms, applied coordinate-wise.*

Proof. sorry □

2.3 Cartan's Criteria and the Structure Theorem

In this section, we will see the usefulness of the Killing Form. Let L be a Lie algebra, and denote by κ the Killing Form on L.

2.3.1 Preliminaries from Linear Algebra

In this subsection, we will prove some important results from Linear Algebra that will prove useful going forward.

Lemma 2.3.1. *Let V be a finite-dimensional $\mathbb{C}$-vector space, and let $x \in \mathfrak{gl}(V)$ be a linear map with Jordan Decomposition $x = d + n$, where d is diagonal and n is nilpotent. Write*

$$d = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

with respect to some Jordan basis, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x and $n = \dim(V)$. Define

$$\overline{d} := \begin{bmatrix} \overline{\lambda_1} & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{bmatrix}$$

to be the diagonal matrix whose entries are the complex conjugates of the eigenvalues of x. If the λ_i-eigenspace $V(d)_{\lambda_i}$ of d is equal to the $\overline{\lambda_i}$-eigenspace $V(\overline{d})_{\overline{\lambda}_i}$ of $\overline{d}$ for all i, then there exists a polynomial $p \in \mathbb{C}[X]$ such that $p(n) = \overline{d}$.

Proof. sorry □

We will also remind the reader of the definition of non-degeneracy for bilinear forms.

Definition 2.3.2 (Degeneracy of a Bilinear Form)**.** Let $(\cdot, \cdot)$ be a bilinear form. We say that $(\cdot, \cdot)$ is **degenerate** if there exists some $x \neq 0$ such that $(x, y) = 0$ for all y.

Definition 2.3.3 (Non-Degeneracy of a Bilinear Form)**.** We say the bilinear form $(\cdot, \cdot)$ is **non-degenerate** if it is not degenerate, ie, if for all $x \neq 0$, there exists a y such that $(x, y) \neq 0$.

Finally, we will note the following result on direct sums and non-degeneracy.

Proposition 2.3.4. *If $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$ are non-degenerate, then so is $(V_1 \oplus V_2, \langle \cdot, \cdot \rangle_{1\oplus 2})$, where $\langle \cdot, \cdot \rangle_{1\oplus 2}$ is defined coordinate-wise.*

Proof. sorry □

2.3.2 Cartan's Criterion for Solvability

In this subsection, we discuss and prove Cartan's Criterion for Solvability, also known as Cartan's First Criterion.

Theorem 2.3.5 (Cartan's First Criterion)**.** *L is solvable if and only if for all $\ell \in L$ and $\ell' \in L'$, $\kappa(\ell, \ell') = 0$, where L' is the derived subalgebra of L.*

One direction is easily proven.

Lemma 2.3.6. *If L solvable, then for all $\ell \in L$ and $\ell' \in L'$, $\kappa(\ell, \ell') = 0$.*

Proof. Assume that L is solvable. Then, $\mathrm{ad}(L) \subset \mathfrak{gl}(L)$ is solvable too: by the First Isomorphism Theorem, $\mathrm{ad}(L) \cong L/\ker(\mathrm{ad})$, and we know that any quotient of a solvable Lie algebra is solvable. Lie's Theorem then tells us that $\mathrm{ad}(L) \subseteq \mathfrak{t}(L)$ (with respect to some basis of L), meaning that $\mathrm{ad}(L') \subset \mathfrak{u}(L)$ (as all the diagonal entries of the commutator of a matrix in L with another matrix in L are zero). sorry □

The converse is a bit more involved, and requires an intermediate proposition.

Proposition 2.3.7. *Let $K \leq \mathfrak{gl}(n)$ be a Lie subalgebra such that the trace form $(A, B) \mapsto \mathrm{Tr}(AB) : K \times K \to \mathbb{C}$ is identically zero on K. Then, K is solvable.*

Proof. We show that K' is nilpotent. This would imply that K' is solvable. Furthermore, we know that K/K' is abelian, and hence, solvable. Therefore, we would be able to conclude that K is itself solvable.

□

We are now ready to prove the converse of Theorem 2.3.5.

Lemma 2.3.8. *If for all $\ell \in L$ and $\ell' \in L'$, $\kappa(\ell, \ell') = 0$, then L is solvable.*

Proof. The high-level idea is to show that L' is nilpotent by applying Engel's Theorem.

sorry □

2.3.3 Cartan's Criterion for Semi-Simplicity

In this subsection, we discuss and prove Cartan's Criterion for Semi-Simplicity, also known as Cartan's Second Criterion.

Theorem 2.3.9 (Cartan's Second Criterion). *L is semi-simple if and only if the Killing Form κ is non-degenerate.*

Proof. ($\implies$) Assume that L is semi-simple and that $L^\perp \neq \{0\}$, where we consider the orthogonal complement with respect to the Killing Form κ_L. We know, from Proposition 2.2.8, that $L^\perp \trianglelefteq L$, and therefore, that $\kappa_{L^\perp} = \kappa_L|_{L^\perp}$. However, by definition of $L^\perp$,

$$\kappa_{L^\perp} = \kappa_L|_{L^\perp} = 0$$

Therefore, Theorem 2.3.5 (Cartan's First Criterion) tells us that L is solvable. However, this tells us that L cannot be semi-simple, a contradiction. Therefore, κ cannot be degenerate.

($\impliedby$) `sorry`

□

2.3.4 The Structure Theorem for Complex, Semi-Simple Lie Algebras

We are now ready for the important Structure Theorem for Complex, semi-simple Lie algebras.

Theorem 2.3.10 (The Structure Theorem for Complex, Semi-Simple Lie Algebras). *L is semi-simple if and only if L is a direct sum of finitely many semi-simple Lie algebras.*

Proof. ($\implies$)

Let $I \trianglelefteq L$ be a nonzero ideal. Pick I to be minimal, in the sense that if any other ideal is properly contained in I, it must be zero. `sorry`

□

2.4 Generalising the Additive Jordan Decomposition to Semi-Simple Lie Algebras

We already know what Jordan Decompositions look like in the context of the general linear Lie algebra: it is simply a special decomposition of a linear map as the sum of a diagonal and a nilpotent linear map. In this section, we will explore a similar notion for elements of semi-simple algebras.

Throughout this section, we will assume that L is semi-simple, and fix an arbitrary element $x \in L$. The main result that we will prove is the following.

Theorem 2.4.1. *There is a unique decomposition*

$$x = d + n \tag{2.4.1}$$

where d is diagonalisable and n is nilpotent. In particular,

1. $\mathrm{ad}(d) \in \mathfrak{gl}(L)$ *is diagonalisable.*
2. $\mathrm{ad}(n) \in \mathfrak{gl}(L)$ *is nilpotent.*
3. $[d, n] = 0$.

We will prove existence and uniqueness separately. We will build on what we know about Jordan Decompositions from Linear Algebra.

Consider the adjoint $\mathrm{ad}(x) \in \mathfrak{gl}(L)$. We know that $\mathrm{ad}(x)$ admits a Jordan decomposition $D + N$, with D diagonalisable and N nilpotent. These D and N satisfy the properties we would expect to have of $\mathrm{ad}(d)$ and $\mathrm{ad}(n)$, where d and n are as in (2.4.1) (if such elements exist). In particular, since $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is linear, we would have

$$\mathrm{ad}(x) = \mathrm{ad}(d + n) = \mathrm{ad}(d) + \mathrm{ad}(n)$$

We will use the same notation as above for the remainder of this section.

2.4.1 Existence

It suffices to show that $\mathrm{ad}(d) = D$ and $\mathrm{ad}(n) = N$. Before we can do this, we will need some machinery about $\mathrm{ad}(L)$, the image of $\mathrm{ad} : L \to \mathfrak{gl}(L)$ in $\mathfrak{gl}(L)$, and $\mathrm{Der}(L)$, the space of all derivations of L (cf. Definition 1.1.42).

A number of the following are not true of when working in arbitrary Lie algebras. Many rely quite heavily on the semi-simplicity of L.

We have already seen, in Lemma 1.1.46, that $\mathrm{ad}(L) \leq \mathrm{Der}(L)$. It turns out that we can show something even stronger in the case of semi-simple Lie algebras.

Lemma 2.4.2. $\mathrm{ad}(L) \trianglelefteq \mathrm{Der}(L)$.

Proof. sorry □

We have a similar (and less nontrivial) result about the relationship between $\mathrm{ad}(L)$ and L.

Lemma 2.4.3. $\mathrm{ad}(L) \trianglelefteq L$.

Lemma 2.4.4. $\mathrm{ad}(L)^{\perp} = \{0\}$, *where we take the orthogonal complement with respect to the Killing Form* κ_L.

The fact that $\mathrm{ad}(L)$ is an ideal of L

This allows us to

Proposition 2.4.5. $\mathrm{Der}(L) = \mathrm{ad}(L)$.

Proof. □

2.4.2 Uniqueness

2.4.3 Properties

Throughout this subsection, fix $\delta \in \mathrm{Der}(L)$. Denote its additive Jordan Decomposition as $\delta = \nu + \mu$. Write

$$L = \bigoplus_{\alpha} L_\alpha$$

Definition 2.4.6. For some $\alpha \in \mathbb{C}$, define

$$L_\alpha := \{v \in L \mid \exists n \in \mathbb{N} \text{ s.t. } (\delta - \alpha \cdot \mathrm{id}_L)^n v = 0\}$$

Lemma 2.4.7. *For some $\alpha, \beta \in \mathbb{C}$, $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.*

Proof. Fix $x \in L_\alpha$ and $y \in L_\beta$. sorry □

Lemma 2.4.8. $\nu, \mu \in \mathrm{Der}(L)$.

Proof. Observe that $\nu|_{L_\alpha} = \alpha \cdot \mathrm{id}_{L_\alpha}$, ie, when restricted to L_α, ν acts as α times the identity. Now, sorry □

2.5 The Representation Theory of $\mathfrak{sl}(2)$

It is said that an understanding of the representation theory of $\mathfrak{sl}(2)$ is commensurate with an understanding of representation theory. We will move forward under this assumption.

2.5.1 Generators and Relations

We begin by describing the generators and relations for $\mathfrak{sl}(2)$. We will fix the notation in this subsection for the entirety of the section.

Define

$$e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.5.1}$$

Proposition 2.5.1 (Generators of $\mathfrak{sl}(2)$). *$\{e, f, h\}$ is generates $\mathfrak{sl}(2)$. That is, their linear span consists of all elements of $\mathfrak{sl}(2)$.*

Proof. sorry ☐

We can now describe the relations between e, f, h.

Proposition 2.5.2 (Relations between the Generators of $\mathfrak{sl}(2)$). *We have the following relations between f, g, h:*

$$[e, f] = h \tag{2.5.2}$$

$$[h, f] = -2f \tag{2.5.3}$$

$$[h, e] = 2e \tag{2.5.4}$$

Proof. sorry ☐

2.5.2 Representations in Terms of Homogeneous Polynomials

Fix $d \in \mathbb{N}$. Consider the vector space

$$V_d := \operatorname{Span}\left(\{X^d, X^{d-1}Y, X^{d-2}Y^2, \ldots, X^2Y^{d-2}, XY^{d-1}, Y^d\}\right)$$

of homogeneous polynomials of degree d over $\mathbb{C}$. Define the maps

$$e' := X\frac{\partial}{\partial Y} \qquad f' := Y\frac{\partial}{\partial X} \qquad h' := X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} \tag{2.5.5}$$

that all lie in $\mathfrak{gl}(V_d)$. We will fix the above notation for the rest of the section.

Proposition 2.5.3. *There exists a unique representation $\phi_d : \mathfrak{sl}(2) \to \mathfrak{gl}(V_d)$ such that $\phi_d(e) = e'$, $\phi_d(f) = f'$, and $\phi_d(h) := h'$.*

Proof. □

The way that Proposition 2.5.3 should be read is that "for every $d \in \mathbb{N}$, there is a unique representation ϕ_d" of the given nature.

We can say even more about the representations ϕ_d.

Theorem 2.5.4. *Each representation ϕ_d is irreducible.*

Proof. □

2.5.3 Classifying All Representations of $\mathfrak{sl}(2)$

It will turn out that every representation of $\mathfrak{sl}(2)$ is isomorphic to some ϕ_d, with the specific d having to do with the dimension of the chosen representation.

First, note the following fact (which we won't bother to prove).

Lemma 2.5.5. $\dim(V_d) = d + 1$.

We will now state a few intermediate lemmas about the eigenvectors of e', f' and h'.

Lemma 2.5.6. *Let v be a non-zero eigenvector of h' with eigenvalue λ, so that $h'v = \lambda v$. Then,*

1. *$e'v$ is an eigenvector of h' with eigenvalue $\lambda + 2$.*
2. *$f'v$ is an eigenvector of h' with eigenvalue $\lambda - 2$.*

Proof. `sorry` □

Lemma 2.5.7. *There exists a non-zero eigenvector of h' such that $e'v = 0$.*

Chapter 3

Root Systems

In this chapter, we explore the concept of root systems. We classify them and use that classification to classify semi-simple Lie algebras.

3.1 Cartan Subalgebras

Throughout this section, let L denote a semi-simple Lie algebra. We begin by defining and studying semi-simple elements of L.

3.1.1 Semi-Simple Elements

Definition 3.1.1 (Semi-Simplicity of Elements)**.** An element $x \in L$ is **semi-simple** if it is equal to the diagonal part of its Jordan Decomposition, as in (2.4.1).

Lemma 3.1.2. *L admits a non-zero semi-simple element.*

Proof. Assume, for contradiction, that non non-zero element of L is semi-simple. Then, by Theorem 2.4.1, for all $x \in L$, $\mathrm{ad}(x)$ is nilpotent. Engel's Theorem then tells us that L is nilpotent, which is a contradiction because L is semi-simple. □

We can now define what a Cartan Subalgebra of L is.

Definition 3.1.3 (Cartan Subalgebra). A **Cartan subalgebra** of L is a Lie subalgebra $H \leq L$ such that

1. Every element of H is semi-simple.
2. H is abelian.
3. H is the maximal subalgebra of L (with respect to inclusion) that satisfies the above two properties.

We will prove important properties of Cartan subalgebras in the next subsection.

3.1.2 Properties of Cartan Subalgebras

First of all, we need to make sure what we are doing is sensible.

Lemma 3.1.4. *L admits a Cartan subalgebra.*

Proof. The zero subalgebra satisfies the necessary properties. □

Now, we show that what we are studying is non-trivial.

Proposition 3.1.5. *Let H be the Cartan subalgebra of L. Then, $H \neq 0$.*

Proof. By Lemma 3.1.2, L admits a non-zero semi-simple element. Then, $\text{Span}(x)$ is abelian. Furthermore, any element of it is semi-simple, as ad is linear. Therefore, `sorry` □

3.1.3 Centralisers and the Cartan Decomposition

Recall the definition of a centraliser.

Definition 3.1.6 (Centraliser of a Lie Subalgebra). For a Lie subalgebra $H \leq L$, we define the **centraliser** of H in L to be

$$C_L(H) := \{x \in L \mid \forall y \in L,\ [x, y] = 0\}$$

We can also define the centraliser of an element.

Definition 3.1.7 (Centraliser of an Element)**.** For some $h \in L$, we define the centraliser of h in L to be

$$C_L(h) := \{x \in L \mid [x, h] = 0\}$$

Lemma 3.1.8. *For any $h \in L$, we have that $C_L(h) = C_L(\mathrm{Span}(h))$.*

Lemma 3.1.9. *Let H be a Lie subalgebra of L such that all of its elements are semi-simple. If $H = C_L(H)$, then H is a Cartan subalgebra of L.*

Proof. We know that elements of $C_L(H)$ commute with all elements of H. Since $H = C_L(H)$, it follows that H is abelian. We also have the assumption that all elements of H are semi-simple. Therefore, the only thing we need to show is maximality.

If K is an abelian subalgebra with semi-simple elements that contains H, then elements of K will commute with all elements of K, including elements of H, meaning that $K \subseteq C_L(H)$. But, by assumption, $C_L(H) = H$. Therefore, we have $H \neq K \subseteq C_L(H) = H$, a contradiction. Hence, no such K can exist, making H maximal. □

What's interesting is that the converse is also true: any Cartan subalgebra is equal to its centraliser. Proving this, though, is significantly more difficult. It motivates us to develop a lot of machinery, which is where we will first encounter the notion of the Cartan Decomposition and the roots of a Lie algebra.

Lemma 3.1.10. *Commuting diagonalisable linear maps from any vector space to itself are simultaneously diagonalisable.*

Proof. This is an easy generalisation of the two-dimensional case. `sorry` □

For the remainder of this subsection, let H be an abelian Lie subalgebra of L consisting of semi-simple elements. Write $H = \mathrm{Span}(a_1, \ldots, a_n)$, where h_i are linearly independent. Since H is abelian, we can simultaneously diagonalise elements of its basis.

Proposition 3.1.11. *L admits a decomposition*

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \tag{3.1.1}$$

where L_β denotes the weight space for the action on L of $\mathrm{ad}|_H$, the adjoint map restricted to H, with weight β. Φ here denotes the set of all $\alpha \in H^*$ such that $L_\alpha \neq 0$.

Proof. sorry □

Observe that in the context of the above Proposition, $L_0 = \{v \in L \mid [v, \cdot] = 0\}$. This is nothing but the centraliser of H in L.

The terms of this decomposition are immensely significant, and have special names.

Definition 3.1.12 (The Cartan Decomposition). The decomposition of L given in (3.1.1) is called the **Cartan decomposition** of L.

For the remainder of this subsection, we will adopt the notation used in Proposition 3.1.11.

Definition 3.1.13 (Roots). The elements of Φ are called the **roots** of L.

We can say something about weights.

Lemma 3.1.14 (Behaviour of Weights). *Fix $\alpha, \beta \in H^*$. Then,*

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.
2. $\alpha + \beta \neq 0 \implies \kappa(L_\alpha, L_\beta) = 0$.
3. $\kappa_{L_0 \times L_0}$ *is non-degenerate.*[a]

[a] This relies on the assumption that L is semi-simple.

Proof. Fix $x \in L_\alpha$ and $y \in L_\beta$.

1. For all $h \in H$, we have

$$\mathrm{ad}(h)([x, y]) =$$

sorry

2. Assume that $\alpha + \beta \neq 0$. Then, there exists $h \in H$ such that $\alpha(h) + \beta(h) \neq 0$. Consider the action of $\mathrm{ad}(h)$ on L_α and L_β. sorry

3. sorry

□

Corollary 3.1.15. *If α is a non-zero weight, then every $x \in L_\alpha$ is nilpotent as an element of L.*

Proof. Fix $x \in L_\alpha$. Because of the existence of the Cartan decomposition, it suffices to show that $\mathrm{ad}(x)^n(L_\beta) = 0$ for all weights β (zero or non-zero). Indeed, observe that

$$\mathrm{ad}(x)(L_\beta) \in [L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$$

where the last inclusion follows from the first point in Lemma 3.1.14. We can do something similar with $\mathrm{ad}(x)^2$, $\mathrm{ad}(x)^3$, and so on. sorry □

We are now ready to prove the coveted converse of Lemma 3.1.9.

Theorem 3.1.16. *If $H \subseteq L$ is a Cartan subalgebra, then $H = C_L(H)$.*

Proof. Since H is a Cartan subalgebra, we know that H is abelian. In particular, that means that H is contained in its centraliser. Therefore, it suffices to show that H *contains* its centraliser. We do this in four steps.

1. There exists $h \in H$ such that $C_L(h)$ is minimal with respect to inclusion.

   The idea is that we can express $C_L(H)$ in the following manner:

$$C_L(H) = \bigcap_{h \in H} C_L(h)$$

Therefore, for the right choice $h, h' \in H$, we have the inclusions

$$\begin{array}{ccccc} & & C_L(H) & & \\ & & \cap & & \\ & & C_L(h) \cap C_L(h') & & \\ & \rotatebox{-45}{$\subsetneq$} & & \rotatebox{45}{$\supsetneq$} & \\ C_L(h) & & & & C_L(h') \\ & \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} & \\ & & L & & \end{array}$$

where what makes the choices 'right' is that the inclusions of the intersection of their centralisers into their respective centralisers are *proper*.

sorry

2. $C_L(h) = C_L(H)$.

3. $C_L(h) = C_L(H)$ is nilpotent.

4. $C_L(H)$ is contained in H as a subalgebra.

□

3.2 Understanding Roots

Throughout this section, let H be a Cartan subalgebra of the semi-simple Lie algebra L. Let

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

denote its Cartan decomposition. We will denote by κ the Killing Form on L.

3.2.1 Subalgebras of L Isomorphic to $\mathfrak{sl}(2)$

We begin by deducing properties of L (and its Cartan decomposition) by finding copies of $\mathfrak{sl}(2)$ inside of it. The point is that we understand $\mathfrak{sl}(2)$ very well through its representations, as seen in Section 2.5.

We begin with a fundamental lemma.

Lemma 3.2.1. *For all $\alpha \in \Phi$, $-\alpha \in \Phi$ as well. In other words, the negative of a root is also a root.*

Proof. This turns out to be a direct application of the second point of Lemma 3.1.14. sorry □

This suggests that there might be certain linear relationships between roots. We will soon discover exactly what those relationships are. In the meantime, we can show the following useful result.

Proposition 3.2.2. *Fix $\alpha \in \Phi$ and some non-zero $x \in L_\alpha$. Then, there exists some non-zero $y \in L_{-\alpha}$ such that $\mathrm{Span}(x, y, [x, y])$ is a Lie subalgebra of L. In particular, this subalgebra is isomorphic to $\mathfrak{sl}(2)$.*

Proof. □

3.2.2 Root Strings and Eigenvalues

Fix $\alpha, \beta \in \Phi$. Assume that $\alpha \neq \pm\beta$.

Definition 3.2.3 (Strings)**.** The **α-string through β** is the subspace

$$M := \bigoplus_{n \in \mathbb{Z}} L_{\beta + n\alpha}$$

of L.

Lemma 3.2.4. *M is a $\mathfrak{sl}(\alpha)$-submodule of L, with respect to the adjoint representation.*

Proof. sorry □

It does look a bit strange to call something a 'string'. The reason for this terminology comes from the following proposition.

Proposition 3.2.5. *sorry*

3.2.3 Consequences of the Cartan Decomposition

In this subsection, we give some useful facts about L that follow from its Cartan decomposition. It can be seen as a 'bag of tricks' that we can reach into whenever we need to prove something about roots.

Lemma 3.2.6. *For all $h_1, h_2 \in H$, we have*

$$\kappa(h_1, h_2) = \sum_{\alpha \in \Phi} \alpha(h_1) \cdot \alpha(h_2)$$

Proof. sorry □

This allows us to show that Φ contains a basis of H^*, a fact reminiscent of the fact that any root system contains a basis of its ambient space.

Corollary 3.2.7. $\mathrm{Span}(\Phi) = H^*$.

Proof. sorry □

Seeing as $\Phi \subset H^*$, the Dual Space (with respect to κ) of the Cartan Subalgebra, we know that every element $\alpha \in \Phi$ is expressible as some map $\kappa(t_\alpha, \cdot)$ for some **unique** $t_\alpha \in H$.

Convention. For any $\alpha \in \Phi$, we denote by

- t_α the unique element of H such that
- h_α the unique element of H such that sorry
- e_α some nonzero element of L_α such that

The fact mentioned above will prove useful for the following proposition.

Proposition 3.2.8. *For all $\alpha \in \Phi$, $x \in L_\alpha$ and $y \in L_{-\alpha}$, we have that*

$$[x, y] = \kappa(x, y) \cdot t_\alpha$$

Proof. sorry □

Indeed, we can compute an explicit formula for $t_{\cdot}$.

Proposition 3.2.9. *For all $\alpha \in \Phi$, we have that*

$$t_\alpha = \frac{h_\alpha}{\kappa(e_\alpha, e_\alpha)} \qquad \text{and} \qquad h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$$

Visit https://thefundamentaltheor3m.github.io/LieAlgebrasNotes/main.pdf for the latest version of these notes. If you have any suggestions or corrections, please feel free to fork and make a pull request to my repository.