Carnegie Mellon University

Can Machines Pack Sphere?

Aarhus Mathematics and AI Workshop

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28 January, 2026

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A Bird's-Eye View of the Sphere Packing Project

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The problem, the solution, and the way to get there

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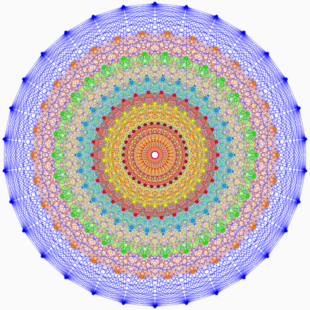

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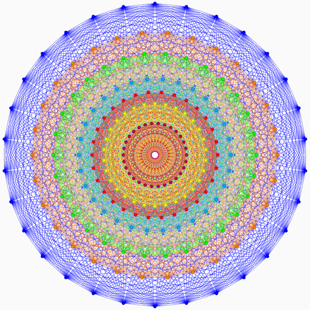

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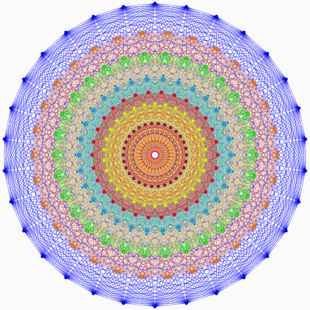

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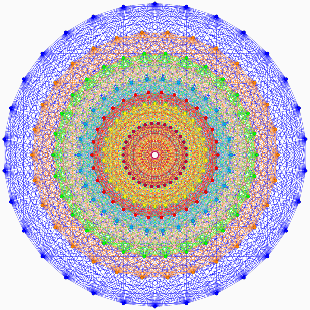

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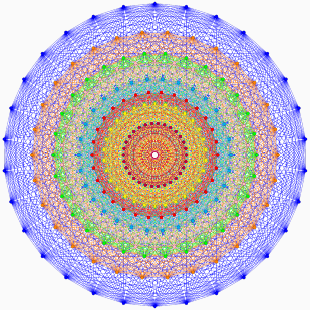

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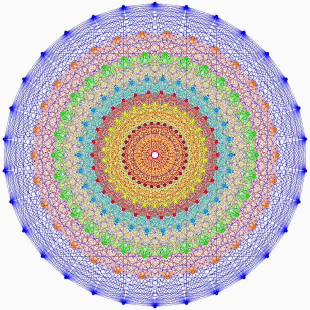

Unfortunately, constructing such functions is extremely difficult. Viazovska's monumental breakthrough was to construct such a function using the theory of modular forms.

A brief history of the sphere packing project

Project Areas

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1. Sphere packing theory

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2. The Cohn-Elkies Linear Programming Bound

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4. Constructing the magic function
5. Proving the modular form inequalities

AI Contributions (So Far)

Gauss: A special case of a more general fact (PR #173)

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```
- theorem perm_I₆ : fourierTransformCLE ℂ (I₆) = I₅ := by sorry
+ theorem perm_I₆ : fourierTransformCLE ℂ (I₆) = I₅ :=
+ by
+   calc
+     fourierTransformCLE ℂ I₆
+         = (SchwartzMap.fourierTransformCLE ℂ).symm I₆ := by
+           ext x
+           have h := congrArg (fun f => f x)
+             (Real.fourierIntegralInv_eq_fourierIntegral_comp_neg
+               (I₆ : EuclideanSpace ℝ (Fin 8) → ℂ))
+           simpa [SchwartzMap.fourierTransformCLE_symm_apply,
+             SchwartzMap.fourierTransformCLE_apply, I₆,
+             schwartzMap_multidimensional_of_schwartzMap_real,
+             SchwartzMap.compCLM_apply] using h.symm
+     _ = I₅ := by
+           simpa [ContinuousLinearEquiv.symm_apply_apply] using
+             (congrArg ((SchwartzMap.fourierTransformCLE ℂ).symm) perm_I₅).symm
```

Initial Gauss commit

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We have a function $I_6 : \mathbb{R}^8 \to \mathbb{R}$ that is radial and Schwartz. Gauss showed that $\widehat{I_6} = I_5$, given that $\widehat{I_5} = I_6$. This is a consequence of a more general fact that for all radial Schwartz functions f, $\widehat{\widehat{f}} = f$.

```diff
+ lemma fourier_involution {V : Type*} [NormedAddCommGroup V] [InnerProductSpace ℝ V]
+     [FiniteDimensional ℝ V] [MeasurableSpace V] [BorelSpace V] {E : Type*} [NormedAddCommGroup E]
+     [NormedSpace ℂ E] [CompleteSpace E] (f : 𝓢(V, E)) :
+     fourierTransformCLE ℂ (fourierTransformCLE ℂ f) = fun x => f (-x) :=
+ by
+   ext x; change 𝓕 (𝓕 f) x = f (-x)
+   simpa [Real.fourierIntegralInv_eq_fourierIntegral_neg, neg_neg] using
+     congrArg (fun g : V → E => g (-x))
+       (f.continuous.fourier_inversion f.integrable ((fourierTransformCLE ℂ) f).integrable)
+
+ lemma radial_inversion {V : Type*} [NormedAddCommGroup V] [InnerProductSpace ℝ V]
+     [FiniteDimensional ℝ V] [MeasurableSpace V] [BorelSpace V] {E : Type*} [NormedAddCommGroup E]
+     [NormedSpace ℂ E] [CompleteSpace E] (f : 𝓢(V, E)) (hf : Function.Even f) :
+     fourierTransformCLE ℂ (fourierTransformCLE ℂ f) = f :=
+ by
+   ext x; simpa [hf x] using congrArg (fun g => g x) (fourier_involution (V:=V) (E:=E) f)
```

```diff
+ -- should use fourier_involution and the radial symmetry of I₅
  theorem perm_I₆ : fourierTransformCLE ℂ (I₆) = I₅ :=
  by
-   calc
-     fourierTransformCLE ℂ I₆
-         = (SchwartzMap.fourierTransformCLE ℂ).symm I₆ := by
-           ext x
-           have h := congrArg (fun f => f x)
-             (Real.fourierIntegralInv_eq_fourierIntegral_comp_neg
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-           simpa [SchwartzMap.fourierTransformCLE_symm_apply,
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-             SchwartzMap.compCLM_apply] using h.symm
-     _ = I₅ := by
-           simpa [ContinuousLinearEquiv.symm_apply_apply] using
-             (congrArg ((SchwartzMap.fourierTransformCLE ℂ).symm) perm_I₅).symm
+   simpa [perm_I₅] using
+     radial_inversion I₅ (fun _ => by
+       simp [I₅, schwartzMap_multidimensional_of_schwartzMap_real, compCLM_apply])
```

Subsequent Gauss commit

Gauss: A bit too trusting of humans? (PR #181)

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We have functions $J_1, \ldots, J_6$ that involve integrals parametrised by $z_1, \ldots, z_6 : [0, 1] \to \mathbb{C}$.

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```
theorem J₃'_smooth' : ContDiff ℝ ∞ RealIntegrals.J₃' := by
  have hJ : RealIntegrals.J₃' = fun x : ℝ => cexp (2 * π * I * x) * RealIntegrals.J₁' x := by
    funext x
    have hEqOn : EqOn (fun t : ℝ => I * ψT' (z₁' t) * cexp (π * I * x * (z₃' t)))
        (fun t : ℝ => cexp (2 * π * I * x) * (I * ψT' (z₁' t) * cexp (π * I * x * (z₁' t))))
        (uIcc (0 : ℝ) 1) := by
      intro t ht
      have ht' : t ∈ Icc (0 : ℝ) 1 := by simpa [uIcc_of_le (by norm_num : (0 : ℝ) ≤ 1)] using ht
      have hz32 : z₃' t = z₁' t + (2 : ℂ) := by
        have h : z₃' t - z₁' t = (2 : ℂ) := by
          simp [z₁'_eq_of_mem ht', z₃'_eq_of_mem ht', one_add_one_eq_two]
        simpa [add_comm] using
          (sub_eq_iff_eq_add' (a := z₃' t) (b := z₁' t) (c := (2 : ℂ))).1 h
      simp [hz32, mul_add, Complex.exp_add, mul_comm, mul_left_comm, mul_assoc]
    simpa [RealIntegrals.J₃', RealIntegrals.J₁'] using
      (intervalIntegral.integral_congr (a := (0 : ℝ)) (b := (1 : ℝ)) hEqOn).trans
        (by
          simp [mul_comm, mul_left_comm, mul_assoc])
  have h_exp : ContDiff ℝ ∞ (fun x : ℝ => cexp (2 * π * I * x)) := by
    have hmul : ContDiff ℝ ∞ (fun x : ℝ => (2 * π * I : ℂ) * (x : ℂ)) :=
      contDiff_const.mul (by simpa using (ofRealCLM.contDiff : ContDiff ℝ ∞ (fun x : ℝ => (x : ℂ))))
    simpa [mul_comm, mul_left_comm, mul_assoc] using hmul.cexp
  simpa [hJ] using (h_exp.mul J₁'_smooth')
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Misformalization in SpherePacking/MagicFunction/a/IntegralEstimates/I6.lean #214

Open

lllvvuu opened last week · edited by pitmonticone

Edits

After `Bound_integrableOn`, put:

```
example : False := by
  have := @Bound_integrableOn
  contrapose! this
  refine' ⟨ -2, _ ⟩;
  obtain ⟨ C₀, hC₀_pos, hC₀ ⟩ := I₆'_bounding_aux_2 ( -2 );
  refine' ⟨ C₀, hC₀_pos, hC₀, _ ⟩;
  norm_num [ mul_assoc, ← Real.exp_add ];
  ring_nf;
  norm_num [ MeasureTheory.integrableOn_const, hC₀_pos.ne' ]
```

This counterexample was found by @Aristotle-Harmonic.

Create sub-issue

gauss-math-inc added a commit that references this issue last week

`This PR contributes a proof for a previously misformalized theorem ...` c330279

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  IntegrableOn (fun t ↦ C₀ * rexp (-2 * π * t) * rexp (-π * r * t)) (Ici (1 : ℝ)) volume := by
  have hb_pos : 0 < π * (r + 2) := mul_pos Real.pi_pos (by linarith)
  have h0 : IntegrableOn (fun t : ℝ => rexp (-(π * (r + 2)) * t)) (Ioi (1 : ℝ)) := by
    simpa using exp_neg_integrableOn_Ioi (a := (1 : ℝ)) (b := π * (r + 2)) hb_pos
  have h1 : IntegrableOn (fun t : ℝ => C₀ * rexp (-(π * (r + 2)) * t)) (Ioi (1 : ℝ)) := h0.const_mul C₀
  have h_eq (t : ℝ) :
      C₀ * rexp (-(π * (r + 2)) * t) =
        C₀ * rexp (-2 * π * t) * rexp (-π * r * t) := by
    have ht : -(π * (r + 2)) * t = (-2 * π * t) + (-π * r * t) := by ring_nf
    simp [ht, Real.exp_add, mul_comm, mul_left_comm]
  have h2 : IntegrableOn (fun t : ℝ => C₀ * rexp (-2 * π * t) * rexp (-π * r * t))
        (Ioi (1 : ℝ)) volume := by
    refine h1.congr_fun ?_ measurableSet_Ioi
    intro t _
    exact h_eq t
  exact
    (integrableOn_Ici_iff_integrableOn_Ioi
        (μ := (volume : Measure ℝ))
        (f := fun t : ℝ => C₀ * rexp (-2 * π * t) * rexp (-π * r * t))
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I took the missing assumption for granted because that r is meant to be the norm of a vector.

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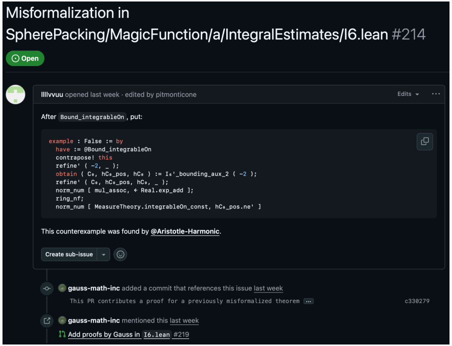

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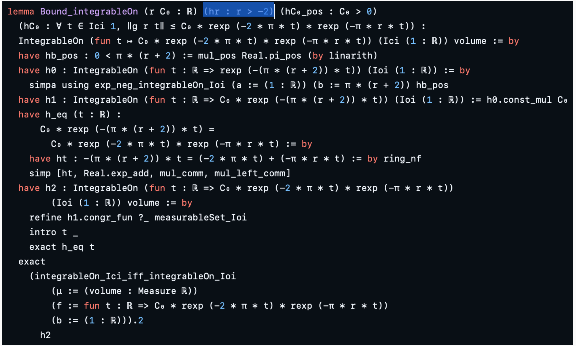

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  have h1 : IntegrableOn (fun t : ℝ => C₀ * rexp (-(π * (r + 2)) * t)) (Ioi (1 : ℝ)) := h0.const_mul C₀
  have h_eq (t : ℝ) :
      C₀ * rexp (-(π * (r + 2)) * t) =
        C₀ * rexp (-2 * π * t) * rexp (-π * r * t) := by
    have ht : -(π * (r + 2)) * t = (-2 * π * t) + (-π * r * t) := by ring_nf
    simp [ht, Real.exp_add, mul_comm, mul_left_comm]
  have h2 : IntegrableOn (fun t : ℝ => C₀ * rexp (-2 * π * t) * rexp (-π * r * t))
        (Ioi (1 : ℝ)) volume := by
    refine h1.congr_fun ?_ measurableSet_Ioi
    intro t _
    exact h_eq t
  exact
    (integrableOn_Ici_iff_integrableOn_Ioi
        (μ := (volume : Measure ℝ))
        (f := fun t : ℝ => C₀ * rexp (-2 * π * t) * rexp (-π * r * t))
        (b := (1 : ℝ))).2
      h2
```

I took the missing assumption for granted because that r is meant to be the norm of a vector. If a human had worked on this proof, I'm sure they'd have spotted it too.

Aristotle: A bit of a sceptic (Issue #214 and PR #219)

Aristotle discovered a misformalisation. Gauss fixed it.

Misformalization in SpherePacking/MagicFunction/a/IntegralEstimates/I6.lean #214

Open

IIllvvuu opened last week · edited by pitmonticone

After `Bound_integrableOn`, put:

```lean
example : False := by
  have := @Bound_integrableOn
  contrapose! this
  refine' ⟨ -2, _ ⟩;
  obtain ⟨ C₀, hC₀_pos, hC₀ ⟩ := I₆'_bounding_aux_2 ( -2 );
  refine' ⟨ C₀, hC₀_pos, hC₀, _ ⟩;
  norm_num [ mul_assoc, ← Real.exp_add ];
  ring_nf;
  norm_num [ MeasureTheory.integrableOn_const, hC₀_pos.ne' ]
```

This counterexample was found by @Aristotle-Harmonic.

Create sub-issue

gauss-math-inc added a commit that references this issue last week

`This PR contributes a proof for a previously misformalized theorem …` c330279

gauss-math-inc mentioned this last week

Add proofs by Gauss in `I6.lean` #219

```lean
lemma Bound_integrableOn (r C₀ : ℝ) (hr : r > -2) (hC₀_pos : C₀ > 0)
  (hC₀ : ∀ t ∈ Ici 1, ‖g r t‖ ≤ C₀ * rexp (-2 * π * t) * rexp (-π * r * t)) :
  IntegrableOn (fun t ↦ C₀ * rexp (-2 * π * t) * rexp (-π * r * t)) (Ici (1 : ℝ)) volume := by
  have hb_pos : 0 < π * (r + 2) := mul_pos Real.pi_pos (by linarith)
  have h0 : IntegrableOn (fun t : ℝ => rexp (-(π * (r + 2)) * t)) (Ioi (1 : ℝ)) := by
    simpa using exp_neg_integrableOn_Ioi (a := (1 : ℝ)) (b := π * (r + 2)) hb_pos
  have h1 : IntegrableOn (fun t : ℝ => C₀ * rexp (-(π * (r + 2)) * t)) (Ioi (1 : ℝ)) := h0.const_mul C₀
  have h_eq (t : ℝ) :
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        (b := (1 : ℝ))).2
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I took the missing assumption for granted because that r is meant to be the norm of a vector. If a human had worked on this proof, I'm sure they'd have spotted it too. However, I seriously doubt their fix would have been to assume $r > -2$...

Claude: Another one joins the fray

Cameron Freer has created some highly effective Lean Skills that have greatly benefitted our project.

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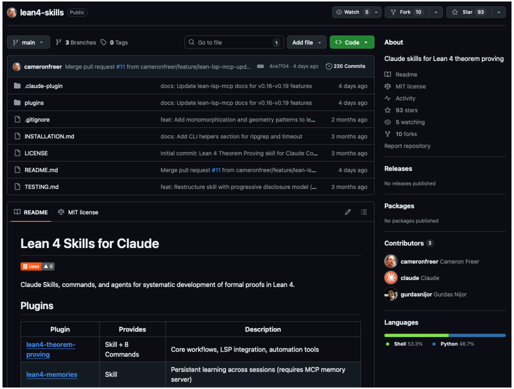

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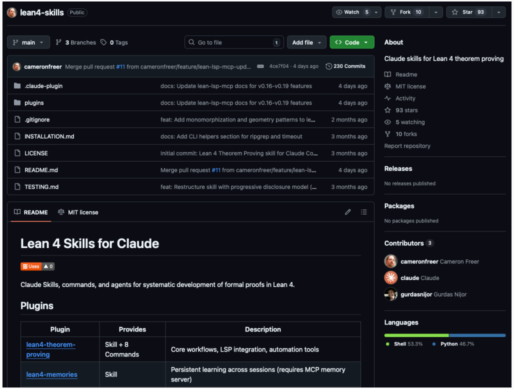

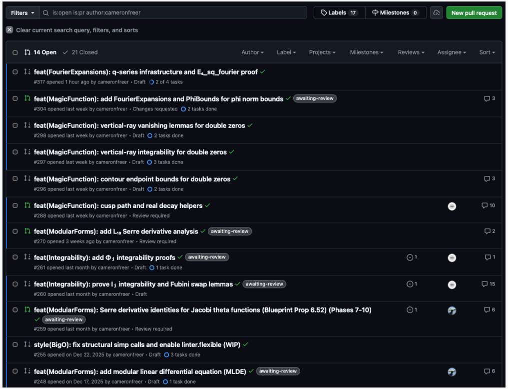

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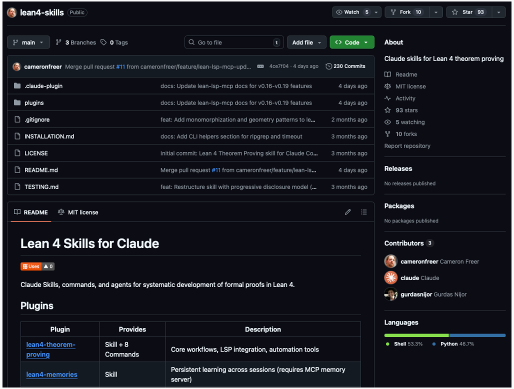
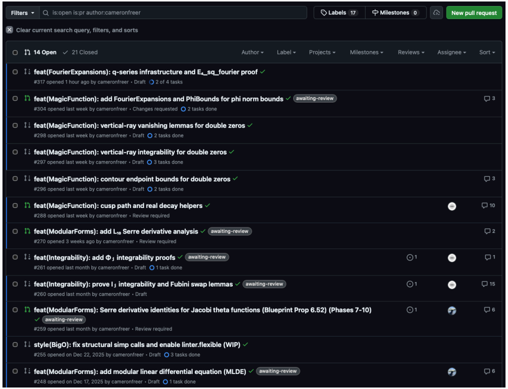
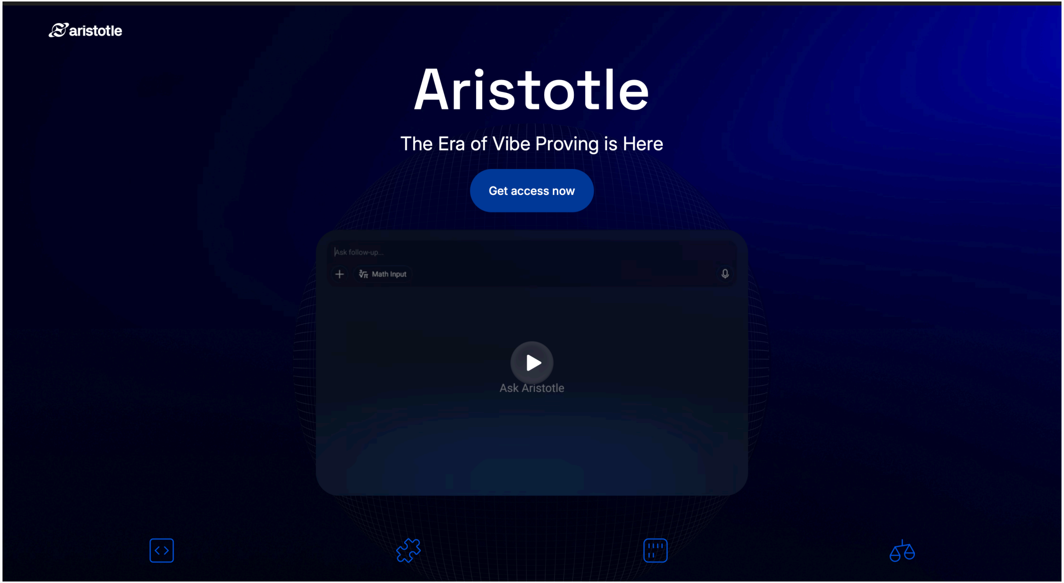


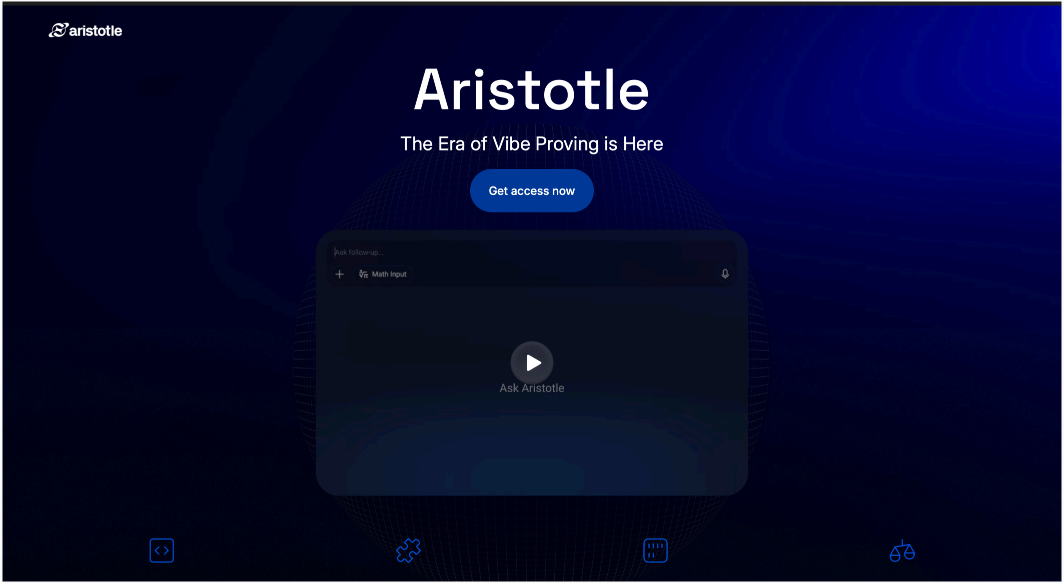


Cameron + Claude have also helped clean up a rather messy (but correct) Aristotle proof.

A Study in Vibe-Proving


aristotle
Aristotle
The Era of Vibe Proving is Here
Get access now
Ask follow-up...
Math Input
Ask Aristotle

Is it though? Let’s find out...



The Challenge (Issue #80)

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We want to show that for all $n \in \mathbb{N}$, $\Delta_n = \Delta_n^{\text{periodic}}$. Necessary for Cohn-Elkies [1, Theorem 3.1].

```
def SpherePackingConstant (d : ℕ) : ℝ≥0∞ := ⨆ S : SpherePacking d, S.density

def PeriodicSpherePackingConstant (d : ℕ) : ℝ≥0∞ := ⨆ S : PeriodicSpherePacking d, S.density

theorem periodic_constant_eq_constant (hd : 0 < d) :
    PeriodicSpherePackingConstant d = SpherePackingConstant d := sorry
```

Proof Strategy

Proof Strategy

alreadydone on Oct 29, 2025 · edited by alreadydone

Edits

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Given an arbitrary packing P of upper density Δ and an arbitrary $\epsilon > 0$, we want to produce a periodic packing with density at least $\Delta - \epsilon$. Given a set S (of nonzero volume), we shall call the quantity $\frac{|S \cap P|}{|S|}$ the density of P in S, where $|S|$ denotes the volume of S.

Choose $\ell > 0$ such that $\frac{(\ell - 4r)^d}{\ell^d} > 1 - \frac{\epsilon}{3}$ (where r is the radius of the balls in P and d is the dimension). Then choose $R_0 > 0$ such that for all $R > R_0$, we have $\frac{(R + \sqrt{d}\ell)^d - (R - \sqrt{d}\ell)^d}{R^d} < \frac{\epsilon}{3}$. By definition of upper density, there exists $R > R_0$ such that the density of P in $B(0, R)$ is at least $\Delta - \frac{\epsilon}{3}$.

The Euclidean space is covered by disjoint hypercubes of side length ℓ of the form $\prod_{i=1}^{d} [n_i \ell, (n_i + 1)\ell)$ with $n_i \in \mathbb{Z}$. Since these hypercubes have diameter $\sqrt{d}\ell$, those that intersect $B(0, R)$ must be contained in $B(0, R + \sqrt{d}\ell)$. Thus the hypercubes contained in $B(0, R + \sqrt{d}\ell)$ cover $B(0, R)$. Similarly, the hypercubes not contained in $B(0, R)$ cannot intersect $B(0, R - \sqrt{d}\ell)$. Denote the union of hypercubes contained in $B(0, R + \sqrt{d}\ell)$ by T and the union of those contained in $B(0, R)$ by S, we then have $S \subseteq B(0, R) \subseteq T \subseteq B(0, R + \sqrt{d}\ell)$ and $T \setminus S \subseteq B(0, R + \sqrt{d}\ell) \setminus B(0, R - \sqrt{d}\ell)$. The density of P in S is $\frac{|S \cap P|}{|S|} \geq \frac{|B(0,R) \cap P| - |B(0,R) \setminus S|}{|B(0,R)|} > (\Delta - \frac{\epsilon}{3}) - \frac{|T \setminus S|}{|B(0,R)|} > (\Delta - \frac{\epsilon}{3}) - \frac{\epsilon}{3}$. Therefore, the density of P in some hypercube $C \subseteq S$ is at least $\Delta - \frac{2\epsilon}{3}$. If a ball in P (thus of diameter $2r$) is not completely contained in C, it cannot intersect the hypercube C' with the same center as C with side length $\ell - 4r$, so excluding these balls from $C \cap P$ decreases the density in C by at most $\frac{|C \setminus C'|}{|C|} = \frac{\ell^d - (\ell - 4r)^d}{\ell^d} < \frac{\epsilon}{3}$, and gives rise to a periodic packing of density $> \Delta - \epsilon$ with C as a fundamental domain.

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alreadydone on Oct 29, 2025 · edited by alreadydone

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alreadydone on Oct 29, 2025 · edited by alreadydone   Edits ···

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- Look at the spheres from the packing contained in this translate.

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alreadydone on Oct 29, 2025 · edited by alreadydone

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Given an arbitrary packing P of upper density Δ and an arbitrary $\epsilon > 0$, we want to produce a periodic packing with density at least $\Delta - \epsilon$. Given a set S (of nonzero volume), we shall call the quantity $\frac{|S \cap P|}{|S|}$ the density of P in S, where $|S|$ denotes the volume of S.

Choose $\ell > 0$ such that $\frac{(\ell - 4r)^d}{\ell^d} > 1 - \frac{\epsilon}{3}$ (where r is the radius of the balls in P and d is the dimension). Then choose $R_0 > 0$ such that for all $R > R_0$, we have $\frac{(R+\sqrt{d}\ell)^d - (R-\sqrt{d}\ell)^d}{R^d} < \frac{\epsilon}{3}$. By definition of upper density, there exists $R > R_0$ such that the density of P in $B(0,R)$ is at least $\Delta - \frac{\epsilon}{3}$.

The Euclidean space is covered by disjoint hypercubes of side length ℓ of the form $\prod_{i=1}^d [n_i\ell, (n_i+1)\ell)$ with $n_i \in \mathbb{Z}$. Since these hypercubes have diameter $\sqrt{d}\ell$, those that intersect $B(0,R)$ must be contained in $B(0,R+\sqrt{d}\ell)$. Thus the hypercubes contained in $B(0,R+\sqrt{d}\ell)$ cover $B(0,R)$. Similarly, the hypercubes not contained in $B(0,R)$ cannot intersect $B(0,R-\sqrt{d}\ell)$. Denote the union of hypercubes contained in $B(0,R+\sqrt{d}\ell)$ by T and the union of those contained in $B(0,R)$ by S, we then have $S \subseteq B(0,R) \subseteq T \subseteq B(0,R+\sqrt{d}\ell)$ and $T \setminus S \subseteq B(0,R+\sqrt{d}\ell) \setminus B(0,R-\sqrt{d}\ell)$. The density of P in S is $\frac{|S\cap P|}{|S|} \ge \frac{|B(0,R)\cap P| - |B(0,R)\setminus S|}{|B(0,R)|} > (\Delta - \frac{\epsilon}{3}) - \frac{|T\setminus S|}{|B(0,R)|} > (\Delta - \frac{\epsilon}{3}) - \frac{\epsilon}{3}$. Therefore, the density of P in some hypercube $C \subseteq S$ is at least $\Delta - \frac{2\epsilon}{3}$. If a ball in P (thus of diameter $2r$) is not completely contained in C, it cannot intersect the hypercube C' with the same center as C with side length $\ell - 4r$, so excluding these balls from $C \cap P$ decreases the density in C by at most $\frac{|C \setminus C'|}{|C|} = \frac{\ell^d - (\ell-4r)^d}{\ell^d} < \frac{\epsilon}{3}$, and gives rise to a periodic packing of density $> \Delta - \epsilon$ with C as a fundamental domain.

👍 1

Proof courtesy Junyan Xu.

- We approximate every sphere packing to arbitrary precision using a periodic packing.
- Pick a clever $\ell > 0$ and consider the scaled lattice $\ell\mathbb{Z}^n$.
- Pick a clever translate of the fundamental domain of $\ell\mathbb{Z}^n$.
- Look at the spheres from the packing contained in this translate.
- The union of those spheres and all their $\ell\mathbb{Z}^n$-translates across $\mathbb{R}^n$ gives the desired packing.

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provided the proofs for them.

I also corrected the statement of `ball_disjoint_inner_of_center_outside` which was false as
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`¬ ball x r ⊆ hypercube ℓ n` instead of `x ∉ innerHypercube ℓ r n`.

The main result `periodic_constant_eq_constant'` is proved using the intermediate lemmas.

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- On one occasion, Aristotle identified a misformalised intermediate lemma and proved it false by providing a counterexample. Claude took a lot longer to realise its mistake, but once it did, it fixed it by adding a missing hypothesis.
- Eventually, both models reached a bottleneck.

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- This is still an active experiment. Let’s see how much longer it takes!

Vibe-Proving Takeaways

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- Different models have different strengths. Using them together might be a good idea.

Concluding Remarks

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It's a new way of working, and the barriers—technical, communication, etc—take some getting used to.

Can machines pack spheres on their own? Not in my experience/not the machines I'm using.

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Can machines pack spheres with guidance?

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Can machines pack spheres with guidance? They're getting there…

Formalising Sphere Packing in Lean

by Chris Birkbeck, Sidharth Hariharan, Gareth Ma, Bhavik Mehta, Seewoo Lee, Maryna Viazovska

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In 2022, Maryna Viazovska was awarded the Fields Medal for solving the sphere packing problem in dimension 8. This project formalises this result in the Lean Theorem Prover following her original paper and followup calculations by Lee. It is an online, open-source collaboration currently being led by Christopher Birkbeck, Sidharth Hariharan, Bhavik Mehta and Seewoo Lee. If you'd like to contribute, you may find the following links useful!

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All links can be found on the project website: `https://thefundamentaltheor3m.github.io/Sphere-Packing-Lean`.

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- All our contributors! (Human and non-human alike)

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Questions?

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