Carnegie Mellon University

The Road to Formalising 8-Dimensional Sphere Packing in Lean

Graduate Student and Postdoc Seminar - Department of Mathematics

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11 September, 2025

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Contents

Formalising Fields Medal-Winning Mathematics

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```
▼LPBound.lean:136:2
▼Tactic state
1 goal
d : ℕ
f : 𝓢(EuclideanSpace ℝ (Fin d), ℂ)
hne_zero : f ≠ 0
hReal : ∀ (x : EuclideanSpace ℝ (Fin d)), ↑(f x).re = f x
hRealFourier : ∀ (x : EuclideanSpace ℝ (Fin d)), ↑(𝓕 (⇑f) x).re = 𝓕 (⇑f) x
hCohnElkies₂ : ∀ (x : EuclideanSpace ℝ (Fin d)), (𝓕 (⇑f) x).re ≥ 0
haux₂ : ∫ (v : EuclideanSpace ℝ (Fin d)), 𝓕 (⇑f) v = ↑(∫ (v : EuclideanSpace ℝ (Fin d)), (𝓕 (⇑f) v).re)
haux₁ : (f 0).re = ∫ (v : EuclideanSpace ℝ (Fin d)), (𝓕 (⇑f) v).re
⊢ 0 < (f 0).re
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Formalising such a recent Fields Medal-winning result would be a monumental breakthrough for the formal theorem proving community.

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4. 13 June, 2025: Viazovska unveils project to public at Big Proof 2025, Cambridge.

Viazovska’s Magical Construction

The problem, the solution, and the way to get there

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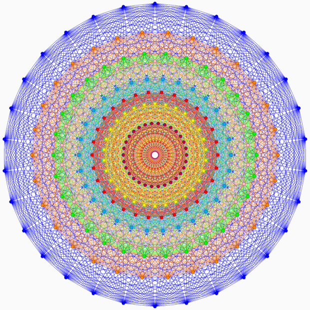

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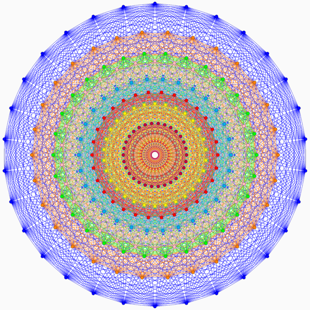

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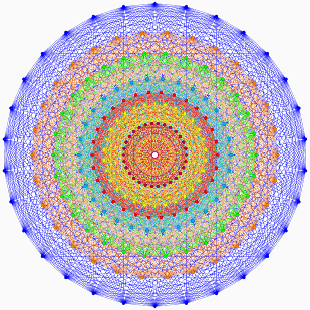

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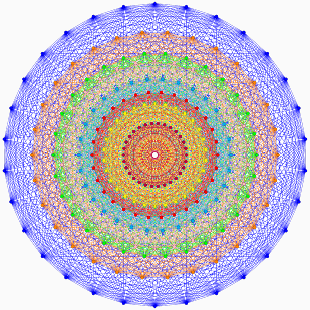

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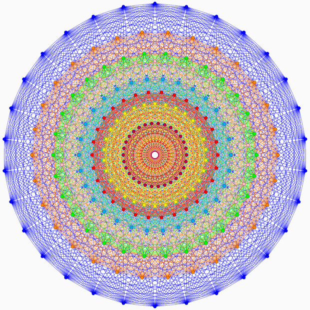

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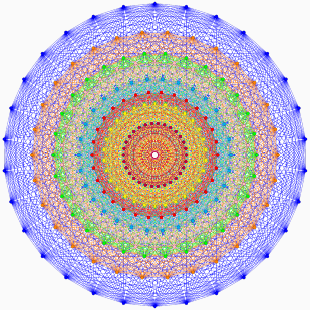

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If f satisfies the above and its upper-bound is optimal, we say f is **magic in $\mathbb{R}^d$**.

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$$a(x) := \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \; e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \; e^{\pi i \|x\|^2 z} \, dz \\ - 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz + 2\int_{i}^{i\infty} \phi_0(z) \; e^{\pi i \|x\|^2 z} \, dz$$

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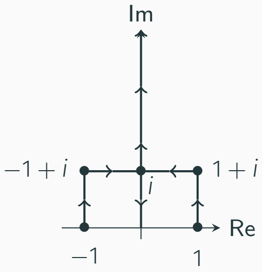

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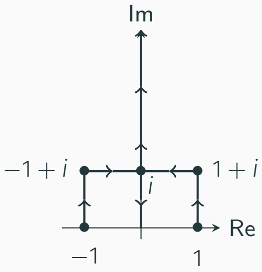

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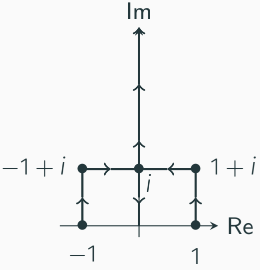
$$g(x) := \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x) \qquad f(x) := g\left(\sqrt{2}x\right)$$



ϕ_0 and ψ_S: the role of modular forms

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$$\alpha(z) = (cz+d)^{-4}\ \alpha\left(\frac{az+b}{cz+d}\right) \qquad \text{for all } z \in \mathbb{H} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

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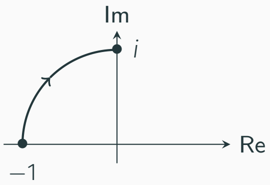
This is crucial to establish that $\widehat{b} = -b$.

Contour Integration, Informally and Formally


Im
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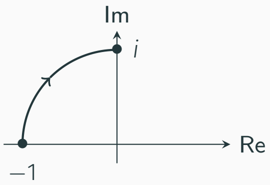
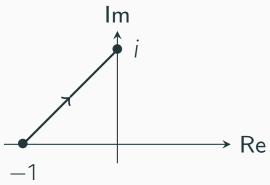
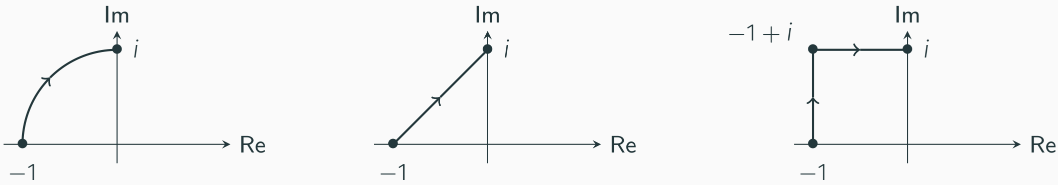
Re
-1

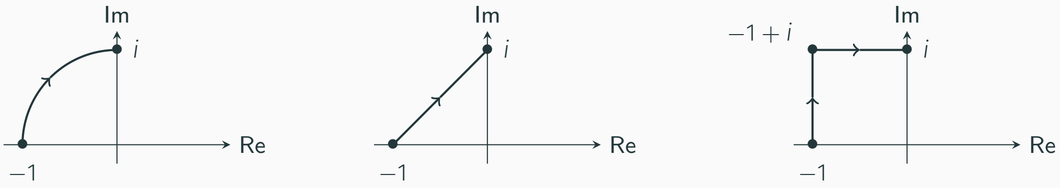
Contour integration






Im
i
Re
-1
Im
i
Re
-1
$-1+i$
Im
i
Re
-1

Contour integration




```
theorem Complex.integral_boundary_rect_eq_zero_of_differentiable_on_off_countable        source
    {E : Type u} [NormedAddCommGroup E] [NormedSpace ℂ E] (f : ℂ → E) (z w : ℂ) (s : Set ℂ)
    (hs : s.Countable) (Hc : ContinuousOn f (Set.uIcc z.re w.re ×ℂ Set.uIcc z.im w.im))
    (Hd :

     ∀ x ∈ Set.Ioo (min z.re w.re) (max z.re w.re) ×ℂ Set.Ioo (min z.im w.im) (max z.im w.im) \ s,
      DifferentiableAt ℂ f x
     )
    :
  (((∫ (x : ℝ) in z.re..w.re, f (↑x + ↑z.im * I)) - ∫ (x : ℝ) in z.re..w.re, f (↑x + ↑w.im * I)) + I •
       ∫ (y : ℝ) in z.im..w.im, f (↑w.re + ↑y * I)) - I • ∫ (y : ℝ) in z.im..w.im, f (↑z.re + ↑y * I)
       = 0
```

Cauchy-Goursat theorem for a rectangle: the integral of a complex differentiable function over the boundary of a rectangle equals zero. More precisely, if `f` is continuous on a closed rectangle and is complex differentiable at all but countably many points of the corresponding open rectangle, then its integral over the boundary of the rectangle equals zero.

Double Zeroes at Lattice Points

$$- 4 \sin^2\left(\frac{\pi \left\|x\right\|^2}{2}\right) \int_0^{i\infty} \phi_0\left(\frac{-1}{z}\right) z^2\, e^{\pi i \left\|x\right\|^2 z}\, dz$$

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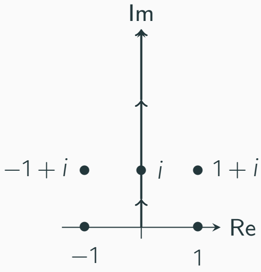

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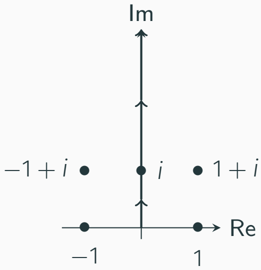
$$-4\sin^2\left(\frac{\pi\|x\|^2}{2}\right)\int_0^{i\infty}\phi_0\left(\frac{-1}{z}\right)z^2\,e^{\pi i\|x\|^2 z}\,dz = \int_{-1}^{-1+i\infty} + \int_1^{1+i\infty} - 2\int_0^{i\infty}$$



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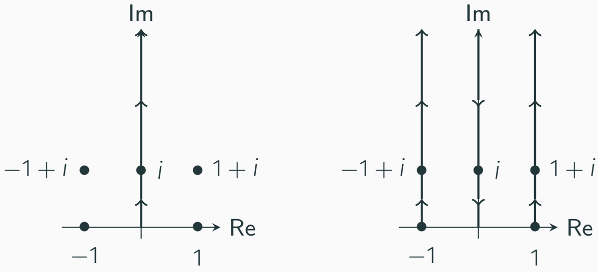

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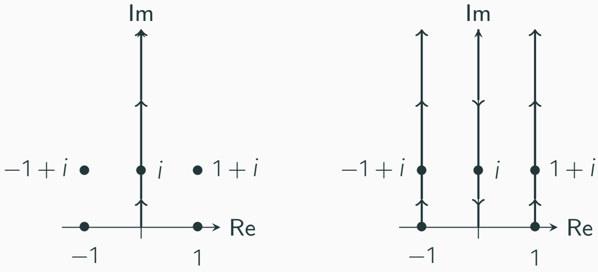

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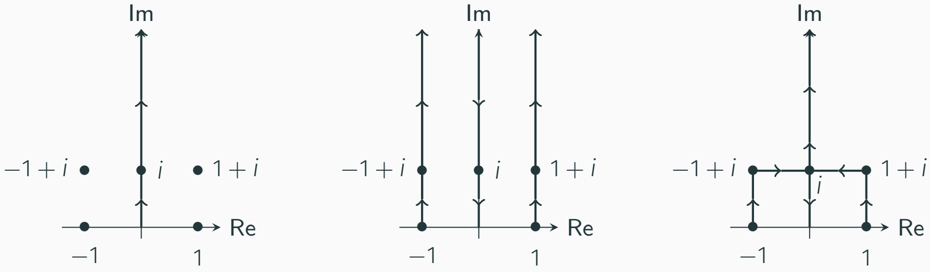

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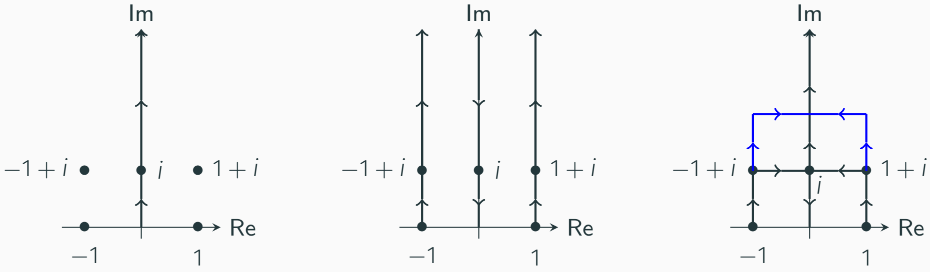

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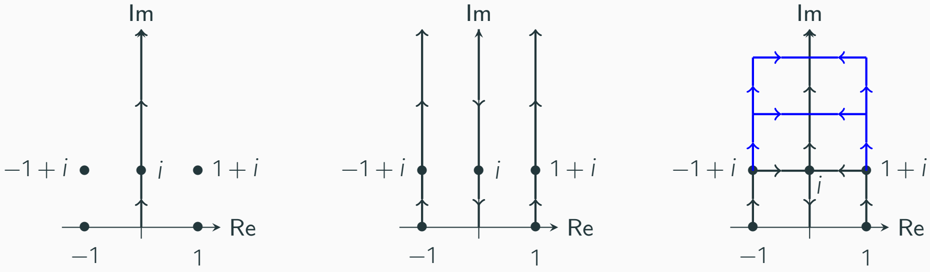

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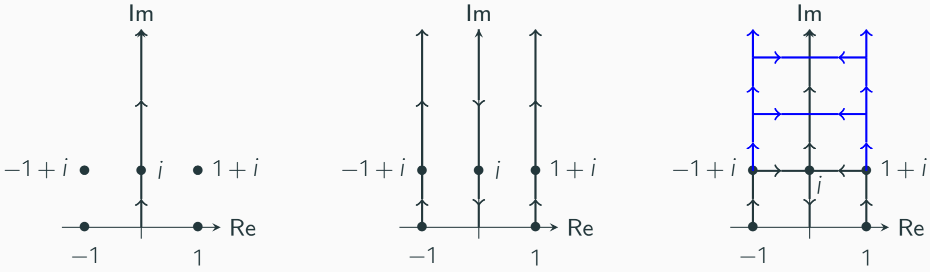

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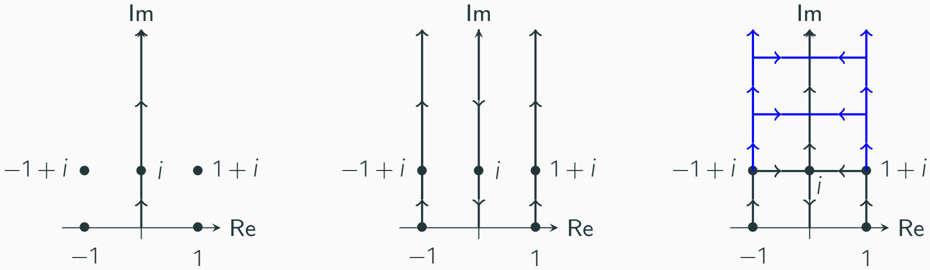
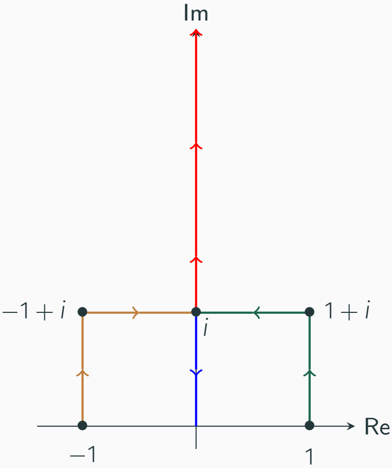
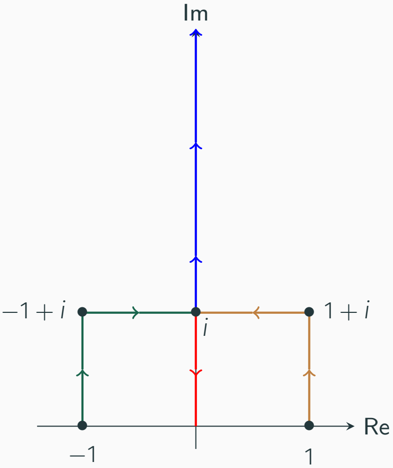


The Fourier Eigenfunction Property

Before taking the Fourier transform:



After taking the Fourier transform:



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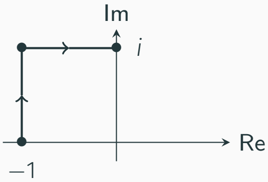

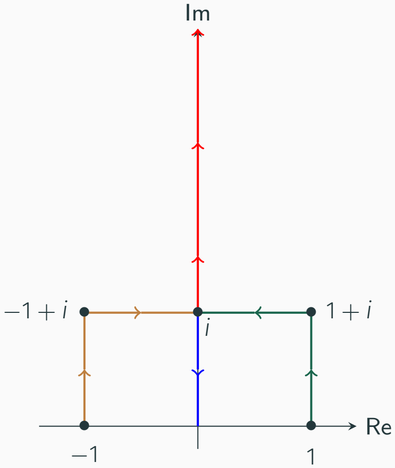

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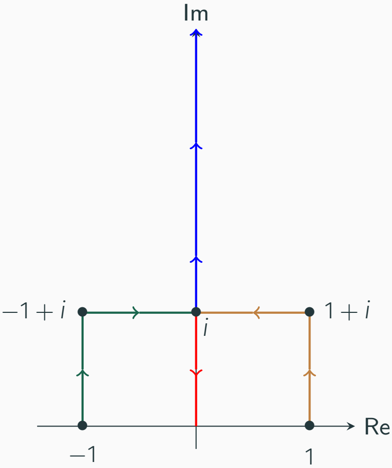

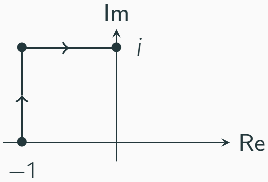

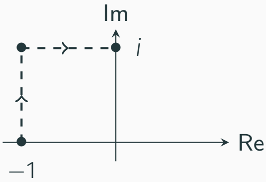

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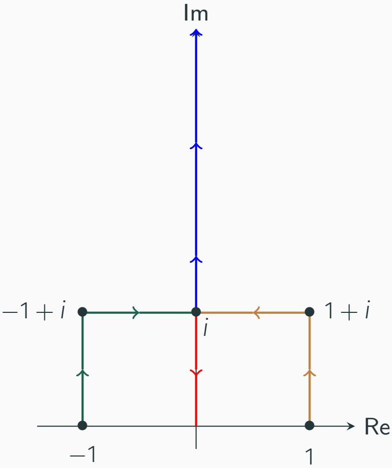

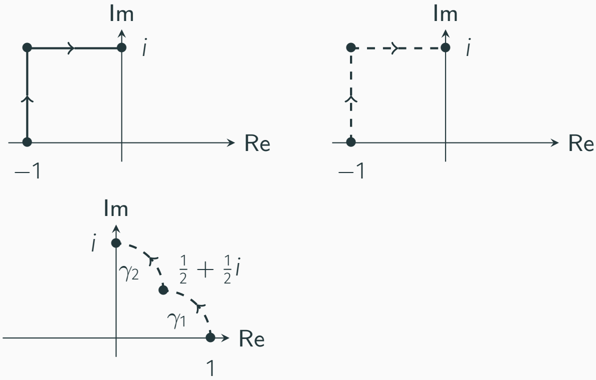

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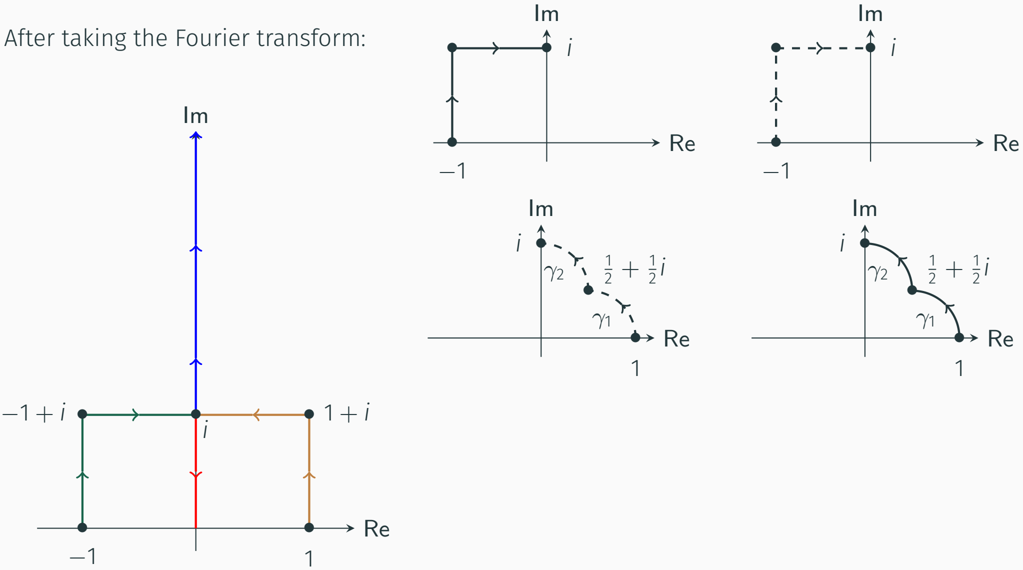

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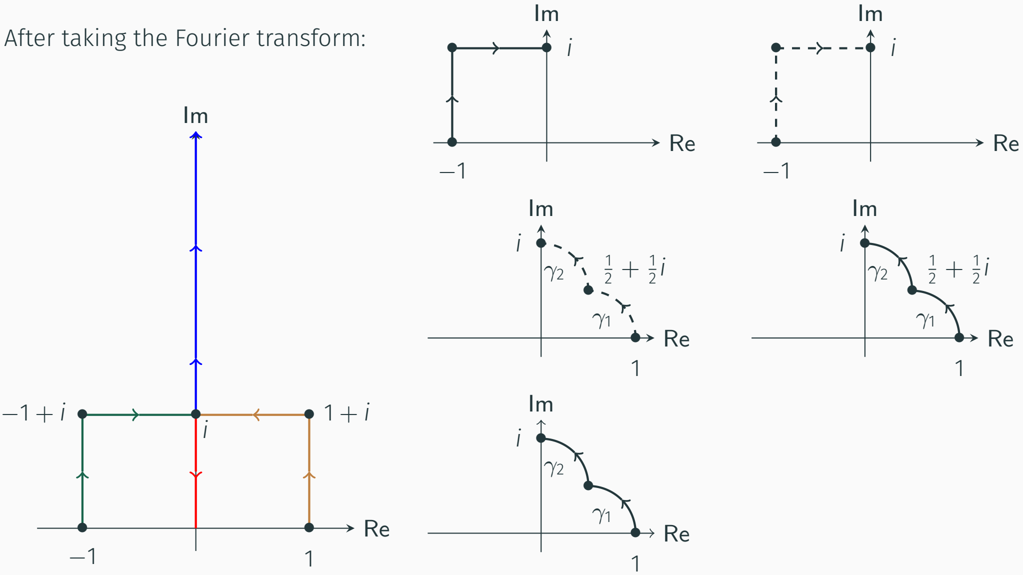

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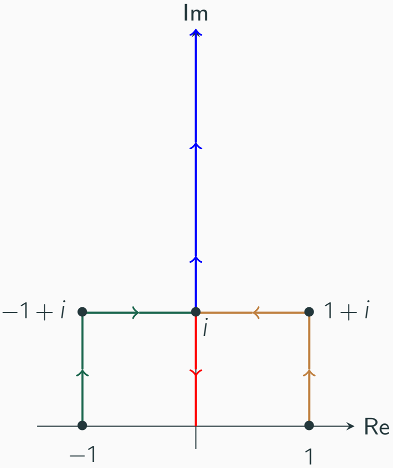

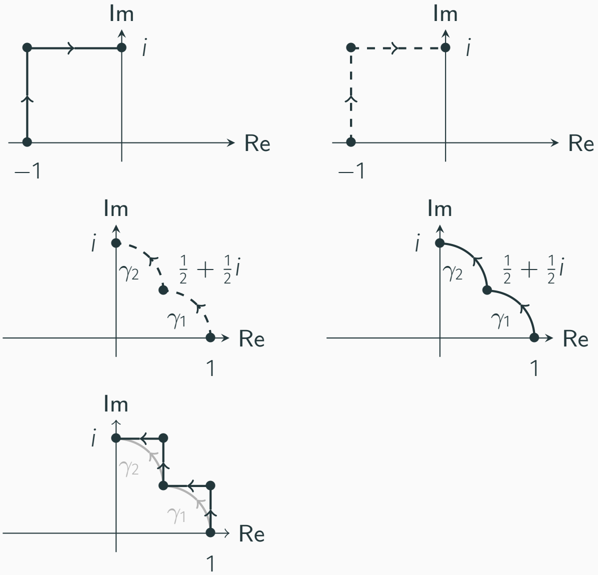

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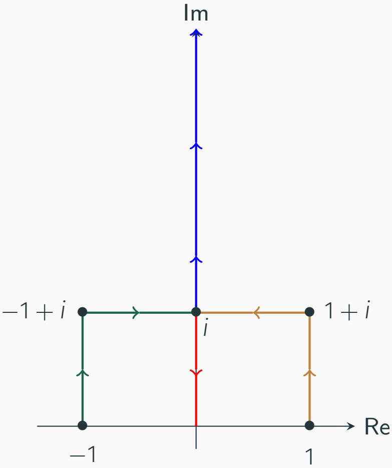

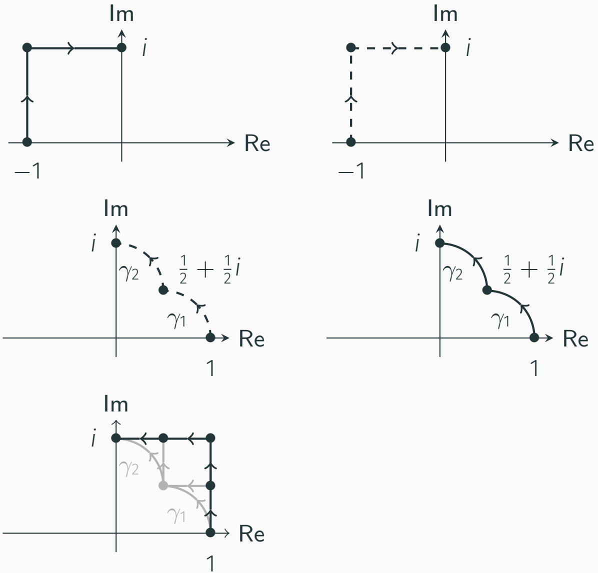

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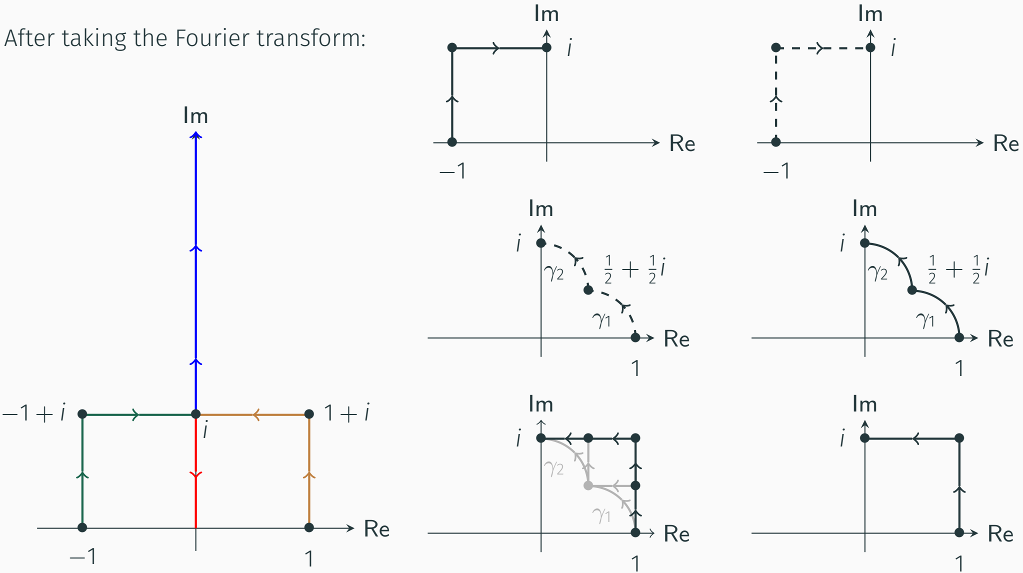

Complex Computations are Complex

Metaprogramming and Complex Numbers

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```
example : (conj (3 + 4 * I) : ℂ) * (3 + 4 * I) = 25 := calc
  _ = (3 - 4 * I) * (3 + 4 * I) := by simp [conj_ofNat, sub_eq_add_neg]
  _ = _ := by
    ring_nf
    simp only [I_sq, neg_mul, one_mul, sub_neg_eq_add]
    norm_cast
    -- The old `norm_num` would also have done the last step but not any of the previous ones
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Questions?

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