Carnegie Mellon University

The Curious Case of Sphere Packing in Lean

Federated Logic Conferences (FLoC)

Sidharth Hariharan
based on joint work with Chris Birkbeck, Seewoo Lee, Gareth Ma, Bhavik Mehta, Maryna Viazovska, and community contributors

25 July, 2026

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Finding our Footing

The Sphere Packing Problem in Dimension 8

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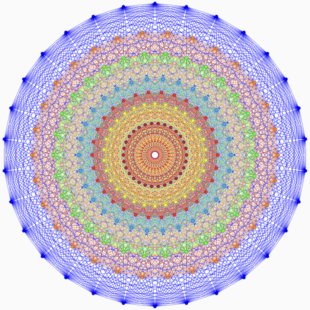

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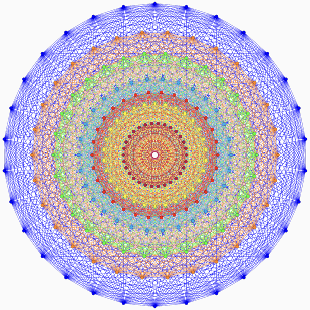

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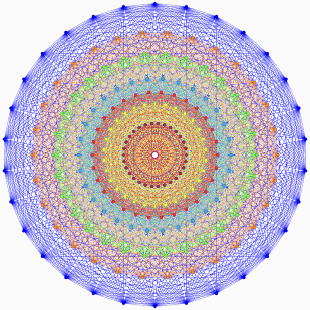

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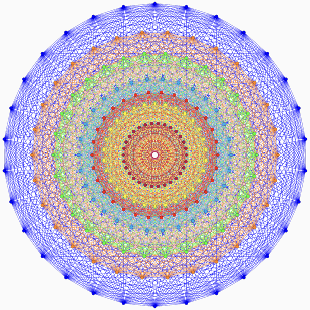

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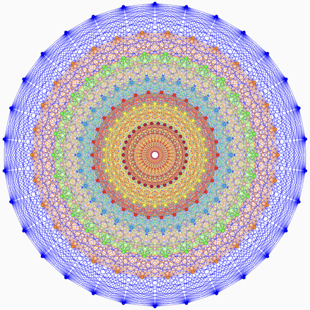

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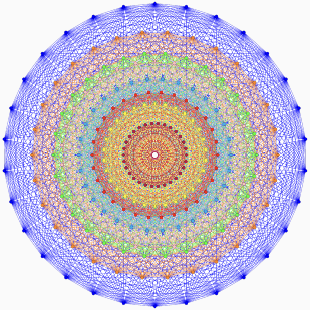

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February 2026 — **Project milestone: formal verification**

Formal Progress

What is a sphere packing, really?

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Here's the definition of a sphere packing from the original paper.

> *Let $X \subseteq \mathbb{R}^d$ be a discrete set of points such that $\|x - y\| \geq 2$ for any distinct $x, y \in X$. Then the union $\mathcal{P} = \bigcup_{x \in X} B_d(x, 1)$ is a sphere packing.*

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def SpherePacking {P : Set V} (hP : Countable P) {r : ℝ} (hr : r > 0)
  (hPr : ∀ p₁ p₂ : V, p₁ ∈ P → p₂ ∈ P → p₁ ≠ p₂ → Euclidean.dist p₁ p₂ > 2*r) : Set V :=
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Second attempt (4 July, 2024):

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  radius : ℝ
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  centers_dist : Pairwise (radius ≤ dist · · : centers → centers → Prop)
structure PeriodicSpherePacking (d : ℕ) extends SpherePacking d where
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Sphere Packing Theory (SH, GM, BM)

A Very Incomplete Summary

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- Definitions of sphere packings, periodic packings, density, sphere packing constant, periodic sphere packing constant
- Density formula for periodic packings
- The theory of the E_8 lattice and the associated sphere packing
- Progress on Cohn-Elkies [2, Theorem 3.1]

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Construction of the Magic Function (SH, BM)

- Definitions and API for a, b, g, their integrands and parametrisations
- Proving `PolyFourierCoeffBound`, a tool to analyse asymptotic behaviour
- Systematically bounding the integrands of a
- Proving slash action relations for ψ_I, ψ_S, ψ_T
- Cauchy-Goursat for unbounded rectangles

A Very Incomplete Summary

Sphere Packing Theory (SH, GM, BM)

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- Density formula for periodic packings
- The theory of the E_8 lattice and the associated sphere packing
- Progress on Cohn-Elkies [2, Theorem 3.1]

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- Serre derivatives: definitions and API, equivariance and invariance
- Serre derivatives of E_2, E_4, E_6
- Defining F and G, restriction to the imaginary axis, positivity of restrictions
- Modular linear differential equations

Metaprogramming

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A `norm_num` extension for $\mathbb{C}$

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```
-- Two atoms, polynomial
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1)) :
    Tendsto (fun z => f z ^ 2 + f z * g z + g
    z ^ 2) atTop (nhds 1) := by tendsto_cont

-- Unused hypotheses don't interfere
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1))
    (unrelated : Tendsto f atBot (nhds 5)) :
    Tendsto (fun z => f z * g z) atTop (nhds
    0) := by tendsto_cont
```

Formal Verification

Formal Verification

Conversation 19 | Commits 18 | Checks 0 | Files changed 287 | +52,187 -15,457

augustepoiroux commented last month • edited Collaborator

Summary

This PR completes the Lean formalization of the theorem that the optimal sphere packing in `ℝ⁸` is the `E₈` lattice packing.
The branch contains the final theorem together with all remaining dependencies, and is checked by the Lean kernel (sorry-free).

Context

The code in this branch was generated by **Gauss** (Math Inc's autoformalization agent) starting from the existing Sphere Packing blueprint and Lean codebase, and then packaged into this PR for review.

Notable changes

- Completes the remaining formalization steps needed for `E₈` optimality in `ℝ⁸`
- Project-wide cleanup / proof golf (including existing files)
- Fixes a sign inconsistency in Prop. 7 (`b(t)`), with the downstream definition of `g` in Thm. 4 updated accordingly
- Migrates to the new module system
- Bumps Lean to `v4.28.0`

Review notes

- We made a best effort to avoid overwriting parts of the codebase that changed independently on `main`. As such, some results may be duplicated.
- ~~The following blueprint-linked Lean statements were commented out in this branch: `ModularForm.dimension_level_one`, `dim_gen_cong_levels`, D_qexp_tsum_pnat, serre_D_two_H₃`~~ -> they have been added back in `38876bc` #diff-ebf1c68f284079a6364607bb7113a4b517ffb358424f42f02a128c225340f25d

🎉 10 🚀 3

Reviewers
- dwrensha
- b-mehta
- seewoo5
- thefundamentaltheor3m
- CBirkbeck
- pitmonticone
- viazovska

At least 1 approving review is required to merge this pull request.

Assignees
No one—assign yourself

Labels
None yet

Projects
None yet

Milestone
No milestone

Development
Successfully merging this pull request may close these issues.
None yet

Finding our Footing, Part 2

Is the project finished?

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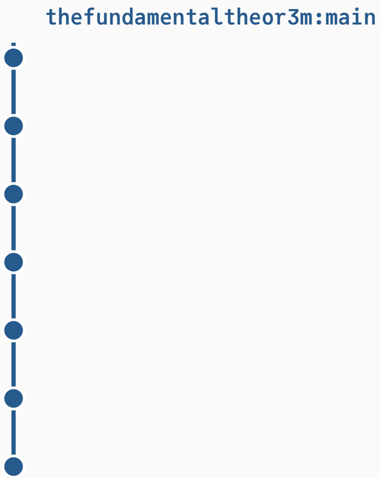

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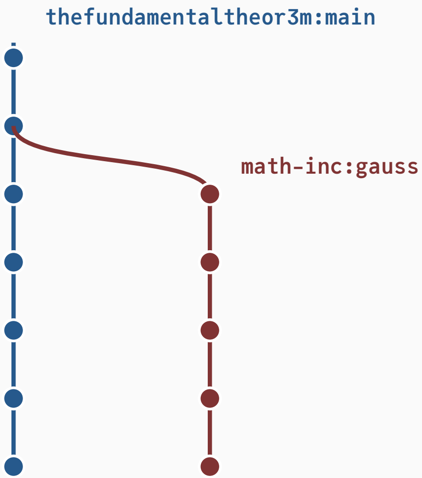

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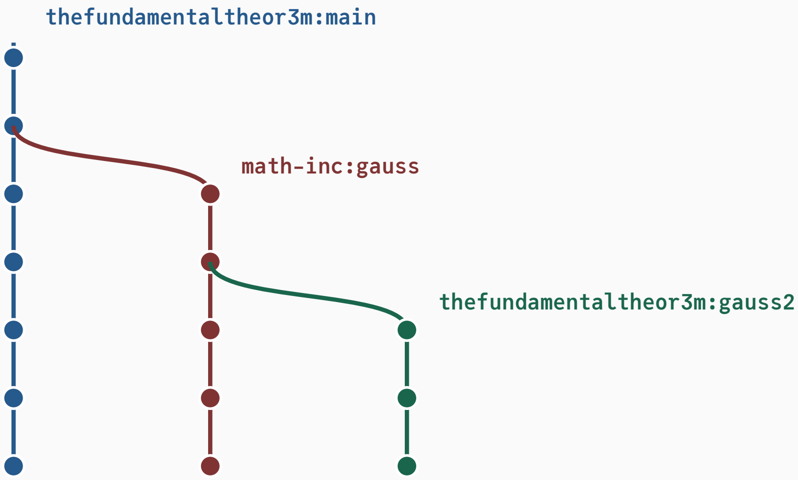

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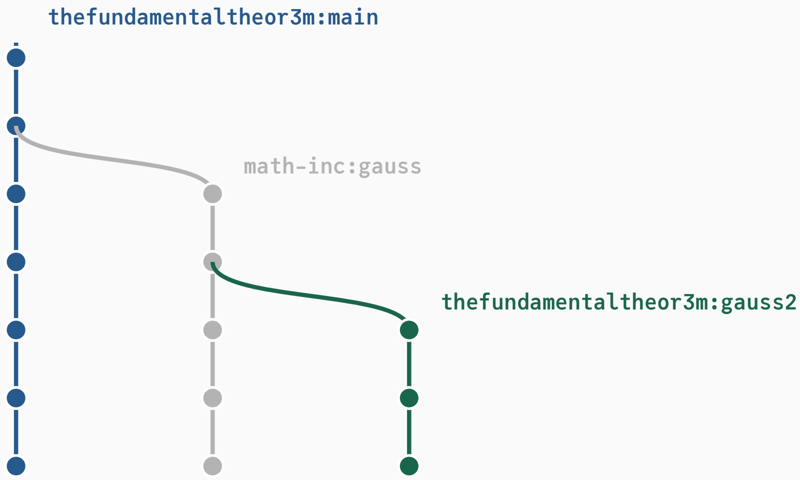

Initial Reviews

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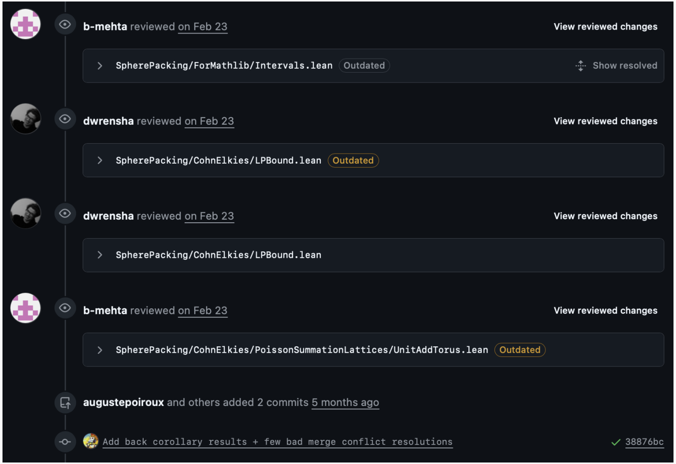

David Renshaw's `TryAtEachStep`

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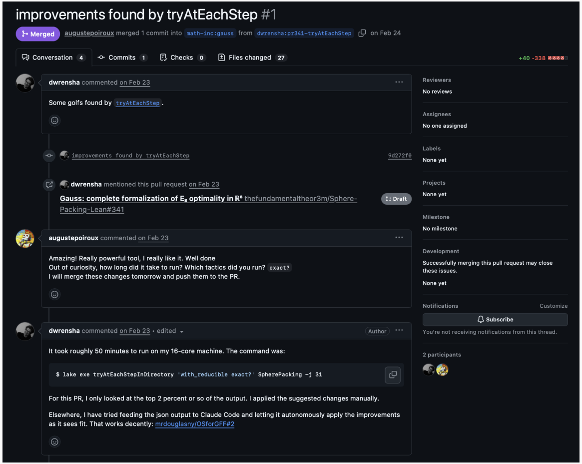

Math, Inc's `AutoGolf`

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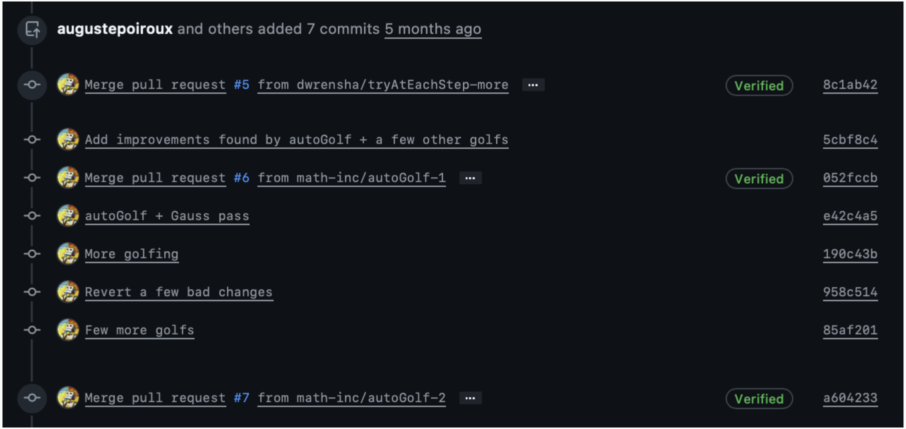
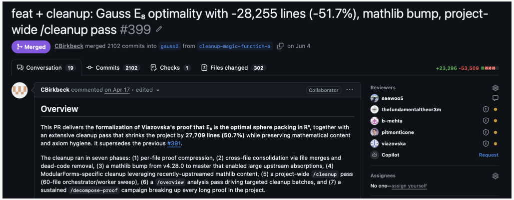
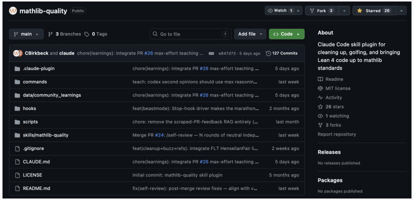


The Many Claudes of Christopher Birkbeck

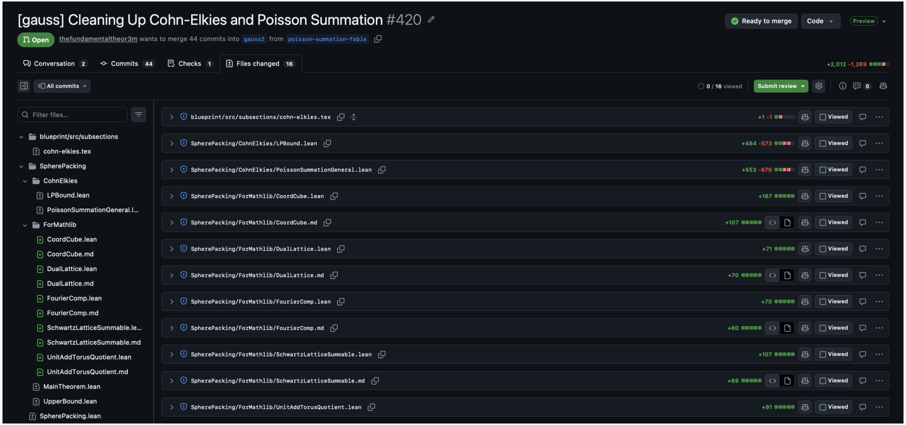


(a) The pull requests



(b) The `mathlib-quality` repository

Targeted Claudeing



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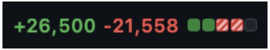

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+26,500 -21,558

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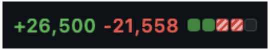

But lines of code isn't everything. The code quality still needs improvement.

In the meantime, progress on `main` has become uncomfortably slow: 45 PRs merged between 23 February 2026 and today as opposed to 150 merged between 1 September 2025 and 31 January 2026.

Looking Ahead

The Virtues of Formalisation

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The eighth (**engineering concepts**), ninth (**communicating**), and tenth (**collaborating**) have been key outcomes.

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We need to continue to be architects of sphere packing.

Maybe we need to go back to `main` and continue doing things by hand.

Acknowledgements

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Thank you very much!

Questions?

[1] J. Avigad.
Mathematics and the formal turn.
arXiv preprint arXiv:2311.00007, 2023.

[2] H. Cohn and N. Elkies.
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