Carnegie Mellon University

Progress in Formalising Sphere Packing in Dimension 8

International Congress on Mathematical Software 2026

Sidharth Hariharan
based on joint work with Chris Birkbeck, Seewoo Lee, Gareth Ma, Bhavik Mehta, Auguste Poiroux, Maryna Viazovska, and community contributors

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> "Formalising mathematics is a great way to learn it."

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February 2026	**Project milestone: formal verification**

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5. Quasimodular forms and modular form inequalities: SL

Key Design Choices

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> *Let $X \subseteq \mathbb{R}^d$ be a discrete set of points such that $\|x - y\| \geq 2$ for any distinct $x, y \in X$. Then the union $\mathcal{P} = \bigcup_{x \in X} B_d(x, 1)$ is a sphere packing.*

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  centers : Set (EuclideanSpace ℝ (Fin d))
  radius : ℝ
  radius_pos : 0 < radius
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structure PeriodicSpherePacking (d : ℕ) extends SpherePacking d where
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  _discrete : DiscreteTopology  := by infer_instance
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$$a(x) := \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \, e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \, e^{\pi i \|x\|^2 z} \, dz$$
$$- 2 \int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz + 2 \int_{i}^{i\infty} \phi_0(z) \, e^{\pi i \|x\|^2 z} \, dz$$

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&- 2\int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz - 2\int_{i}^{i\infty} \psi_S(z) \, e^{\pi i \|x\|^2 z} \, dz
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a and b have three very special properties: they are **radial and Schwartz**, they are (respectively) **Fourier ± 1-eigenfunctions**, and they have **double zeroes at E_8 lattice points**.

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a(x) := &\int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \, e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \, e^{\pi i \|x\|^2 z} \, dz \\
&- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz + 2\int_{i}^{i\infty} \phi_0(z) \, e^{\pi i \|x\|^2 z} \, dz \\
b(x) := &\int_{-1}^{i} \psi_S\left(\frac{-1}{z+1}\right) (z+1)^2 \, e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \psi_S\left(\frac{-1}{z-1}\right) (z-1)^2 \, e^{\pi i \|x\|^2 z} \, dz \\
&- 2\int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz - 2\int_{i}^{i\infty} \psi_S(z) \, e^{\pi i \|x\|^2 z} \, dz \\
g(x) := &\frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x)
\end{aligned}
$$

a and b have three very special properties: they are **radial and Schwartz**, they are (respectively) **Fourier ± 1-eigenfunctions**, and they have **double zeroes at E_8 lattice points**.

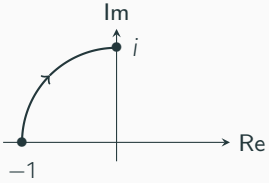
The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a very special function that is crucial to Viazovska's argument. g is defined as a linear combination of two functions a and b, which are defined in terms of quasimodular forms.

$$\begin{aligned} a(x) := &\int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \, e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \, e^{\pi i \|x\|^2 z} \, dz \\ &- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz + 2\int_{i}^{i\infty} \phi_0(z) \, e^{\pi i \|x\|^2 z} \, dz \\ b(x) := &\int_{-1}^{i} \psi_S\left(\frac{-1}{z+1}\right) (z+1)^2 \, e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \psi_S\left(\frac{-1}{z-1}\right) (z-1)^2 \, e^{\pi i \|x\|^2 z} \, dz \\ &- 2\int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz - 2\int_{i}^{i\infty} \psi_S(z) \, e^{\pi i \|x\|^2 z} \, dz \\ g(x) := &\frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x) \end{aligned}$$

a and b have three very special properties: they are **radial and Schwartz**, they are (respectively) **Fourier ± 1-eigenfunctions**, and they have **double zeroes at E_8 lattice points**. This endows g with unique, 'magic' properties.

Choosing Contours

Choosing Contours



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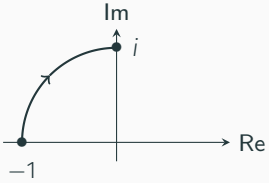

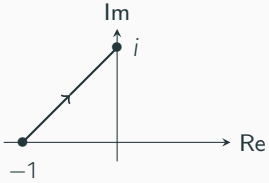

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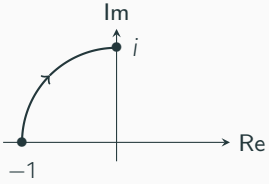

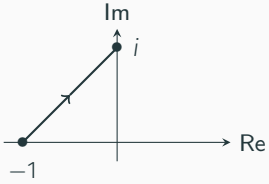

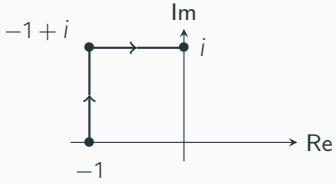

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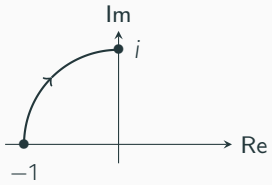

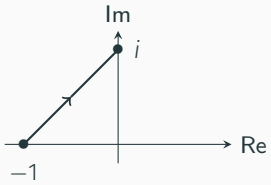

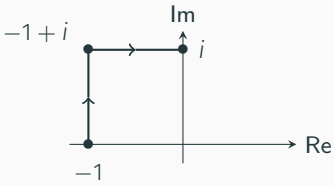

theorem `Complex.integral_boundary_rect_eq_zero_of_differentiable_on_off_countable` source

```
    {E : Type u} [NormedAddCommGroup E] [NormedSpace ℂ E] (f : ℂ → E) (z w : ℂ) (s : Set ℂ)
    (hs : s.Countable) (Hc : ContinuousOn f (Set.uIcc z.re w.re ×ℂ Set.uIcc z.im w.im))
    (Hd :

     ∀ x ∈ Set.Ioo (min z.re w.re) (max z.re w.re) ×ℂ Set.Ioo (min z.im w.im) (max z.im w.im) \ s,
      DifferentiableAt ℂ f x
     )
    :
(((∫ (x : ℝ) in z.re..w.re, f (↑x + ↑z.im * I)) - ∫ (x : ℝ) in z.re..w.re, f (↑x + ↑w.im * I)) + I •
    ∫ (y : ℝ) in z.im..w.im, f (↑w.re + ↑y * I)) - I • ∫ (y : ℝ) in z.im..w.im, f (↑z.re + ↑y * I)
    = 0
```

Cauchy-Goursat theorem for a rectangle: the integral of a complex differentiable function over the boundary of a rectangle equals zero. More precisely, if `f` is continuous on a closed rectangle and is complex differentiable at all but countably many points of the corresponding open rectangle, then its integral over the boundary of the rectangle equals zero.

Defining the Magic Function in Lean

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```
def ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp (π * I * r * (z : ℂ))
                                    ⋮
def ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp (π * I * r * (z : ℂ))
```

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```

```
def ₁ : ℝ → ℂ := fun t ↦ I * ₁' r (z₁' t)
                    ⋮
def ₆ : ℝ → ℂ := fun t ↦ I * ₆' r (z₆' t)
```

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                    ⋮
def ₆ : ℝ → ℂ := fun t ↦ I * ₆' r (z₆' t)
```

```
def I₁' : ℝ → ℂ := fun x ↦ ∫ t in (0 : ℝ)..1, ₁ x t
                    ⋮
def I₆' : ℝ → ℂ := fun x ↦ 2 * ∫ t in Ici (1 : ℝ), ₆ x t
```

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```
def ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp (π * I * r * (z : ℂ))
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```
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```

```
def I₁ : V → ℂ := fun x ↦ I₁' (‖x‖ ^ 2)
                    ⋮
def I₆ : V → ℂ := fun x ↦ I₆' (‖x‖ ^ 2)
```

Defining the Magic Function in Lean

```
def φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp (π * I * r * (z : ℂ))
                                    ⋮
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```

```
def Φ₁ : ℝ → ℂ := fun t ↦ I * φ₁' r (z₁' t)
                                    ⋮
def Φ₆ : ℝ → ℂ := fun t ↦ I * φ₆' r (z₆' t)
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```
def I₁' : ℝ → ℂ := fun x ↦ ∫ t in (0 : ℝ)..1, Φ₁ x t
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```

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def I₁ : V → ℂ := fun x ↦ I₁' (‖x‖ ^ 2)
                                    ⋮
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```

```
def a' : ℝ → ℂ := fun x ↦ I₁' x + I₂' x + I₃' x + I₄' x +
    I₅' x + I₆' x
```

Defining the Magic Function in Lean

```
def ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp (π * I * r * (z : ℂ))
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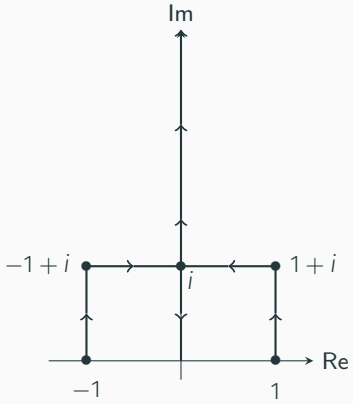

Outcomes and Milestones

Modularity

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Sphere Packing Theory (SH, GM, BM)

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- Definitions of sphere packings, periodic packings, density, sphere packing constant, periodic sphere packing constant
- Density formula for periodic packings
- The theory of the E_8 lattice and the associated sphere packing
- Progress on Cohn-Elkies [2, Theorem 3.1]

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Modular Forms (CB, SL)

- Fixing the definition of E_2
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- H-functions (4th powers of Jacobi Θ-functions) as modular forms
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- Definitions and API for a, b, g, their integrands and parametrisations
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Quasimodular Forms and Inequalities (SL)

- Serre derivatives: definitions and API, equivariance and invariance
- Serre derivatives of E_2, E_4, E_6
- Defining F and G, restriction to the imaginary axis, positivity of restrictions
- Modular linear differential equations

Metaprogramming

A `norm_num` extension for $\mathbb{C}$

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```
example : (1 + 3.5 + I) * (1 + I) = 7 / 2 +
    11 / 2 * I := by norm_num1
example : (3 + 4 * I)⁻¹ * (3 + 4 * I) = 1 :=
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example : 1 + I ≠ 0 := by norm_num1
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example : 10 * re ((1 + 3 * I)⁻¹) = 1 := by
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example : (conj (3 + 4 * I) : ℂ) * (3 + 4 *
    I) = 25 := by norm_num1
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Proving `Tendsto` given atomic limits

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- Proves `Tendsto (fun z => expr(`f_1 `z, ..., `f_n `z)) l (nhds c)` where `Tendsto` f_i `l (nhds` a_i`)` are known and the expression is continuous

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```
-- Two atoms, polynomial
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1)) :
    Tendsto (fun z => f z ^ 2 + f z * g z + g
    z ^ 2) atTop (nhds 1) := by tendsto_cont

-- Unused hypotheses don't interfere
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1))
    (unrelated : Tendsto f atBot (nhds 5)) :
    Tendsto (fun z => f z * g z) atTop (nhds
    0) := by tendsto_cont
```

Formal Verification

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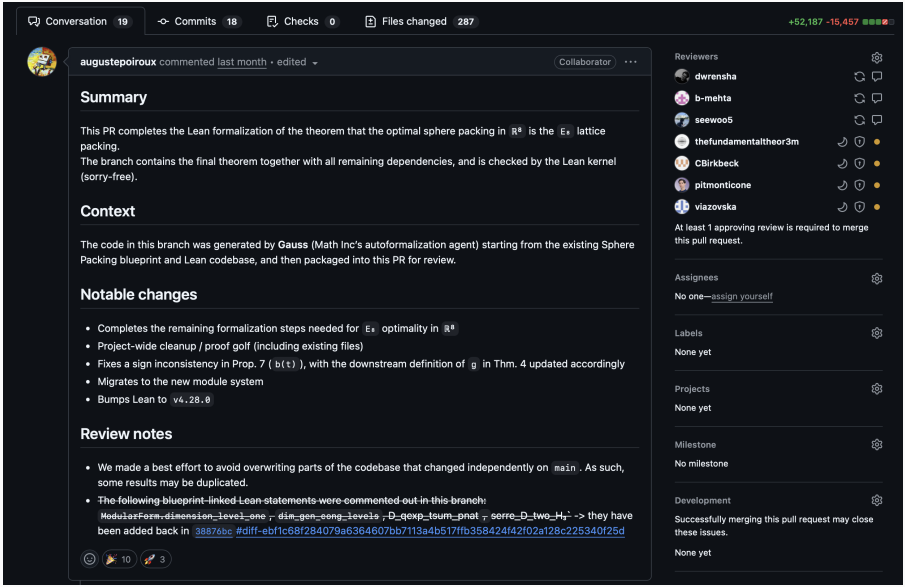

Conversation 19 · Commits 18 · Checks 0 · Files changed 287 · +52,187 -15,457

augustepoiroux commented last month • edited — Collaborator

Summary

This PR completes the Lean formalization of the theorem that the optimal sphere packing in $\mathbb{R}^8$ is the E_8 lattice packing.
The branch contains the final theorem together with all remaining dependencies, and is checked by the Lean kernel (sorry-free).

Context

The code in this branch was generated by **Gauss** (Math Inc's autoformalization agent) starting from the existing Sphere Packing blueprint and Lean codebase, and then packaged into this PR for review.

Notable changes

- Completes the remaining formalization steps needed for E_8 optimality in $\mathbb{R}^8$
- Project-wide cleanup / proof golf (including existing files)
- Fixes a sign inconsistency in Prop. 7 (`b(t)`), with the downstream definition of `g` in Thm. 4 updated accordingly
- Migrates to the new module system
- Bumps Lean to `v4.28.0`

Review notes

- We made a best effort to avoid overwriting parts of the codebase that changed independently on `main` . As such, some results may be duplicated.
- ~~The following blueprint-linked Lean statements were commented out in this branch:~~
  ~~`ModularForm.dimension_level_one`~~ , ~~`dim_gen_cong_levels`~~ , ~~`D_qexp_tsum_pnat`~~ , ~~`serre_D_two_H`~~ -> they have been added back in 3887ebc #diff-ebf1c68f284079a6364607bb7113a4b517ffb358424f42f02a128c225340f25d

🎉 10 · 🚀 3

Reviewers
- dwrensha
- b-mehta
- seewoo5
- thefundamentaltheor3m
- CBirkbeck
- pitmonticone
- viazovska

At least 1 approving review is required to merge this pull request.

Assignees
No one—assign yourself

Labels
None yet

Projects
None yet

Milestone
No milestone

Development
Successfully merging this pull request may close these issues.
None yet

Is the project finished?

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NO!

Remaining Work

The Virtues of Formalisation

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The eighth (**engineering concepts**), ninth (**communicating**), and tenth (**collaborating**) have been key outcomes.

The sixteenth (**using automated reasoning**) and seventeenth (**using artificial intelligence**) have both been greatly impactful to this project, and will remain a big part of its future.

Towards a Formalisation

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There is a lot more riding on the manual approach than we anticipated.

We're just going to have to roll up our sleeves and get it done.

Acknowledgements

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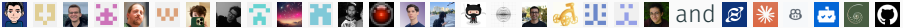

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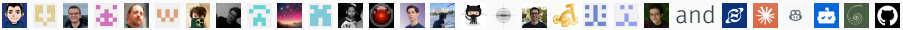

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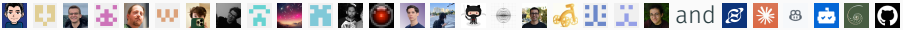

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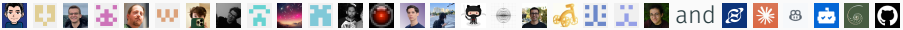

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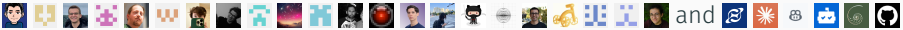

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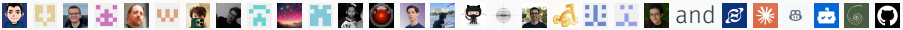

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Thank you very much! Merci beaucoup!

Questions?

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