Carnegie Mellon University

Large-Scale Formalisation: A Student's Perspective

Thematic Program on AI and Mathematics, Korea Institute for Advanced Study

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based on joint work with Chris Birkbeck, Seewoo Lee, Gareth Ma, Bhavik Mehta, Maryna Viazovska, and community contributors

13 July, 2026

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The Beginnings of a Formalisation Project

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"Formalising mathematics is a great way to learn it."

I like learning mathematics. I soon discovered that meeting the demands of interactive theorem provers adds very interesting perspectives.

How a Project Usually Goes

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4. As the formalisation progresses, the blueprint and dependency graph are typically updated, refined, refactored and, if needed, corrected.
5. In parallel, tools and mechanisms are sometimes developed to ease the traversal.

* Traversing the dependency graph is not always the only goal of a formalisation project. An often unspoken goal is to explore the possibility of extending, generalising, reframing, or otherwise modifying the mathematics to make the code more suitable for maintenance and reuse.

Writing a Blueprint

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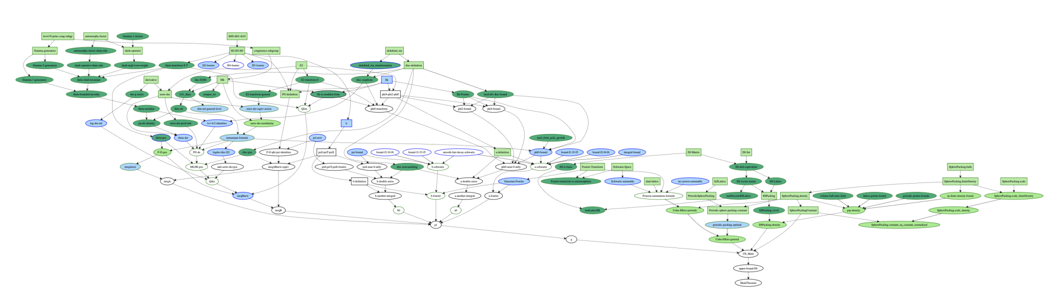

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`https://thefundamentaltheor3m.github.io/Sphere-Packing-Lean/blueprint`

Learning through Blueprinting: Constructing the ‘Magic Function’

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- **June 2025** — **I have a detailed, formalisation-friendly exposition of the construction of the magic function**

The Devil is in the Details

What is a sphere packing, really?

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Here's the definition of a sphere packing from the original paper.

> *Let $X \subseteq \mathbb{R}^d$ be a discrete set of points such that $\|x - y\| \geq 2$ for any distinct $x, y \in X$. Then the union $\mathcal{P} = \bigcup_{x \in X} B_d(x, 1)$ is a sphere packing.*

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def Centres (X : Set V) : Prop := (Discrete d X) ∧ ( ∀ x ∈ X, ∀ y ∈ X, x ≠ y → Euclidean.dist
     x y ≥ 2)
def Packing (X P : Set V) : Prop := (Centres d X) ∧ (P = ⋃ x ∈ X, {y | Euclidean.dist x y <
     1})
def isPacking (P : Set V) : Prop := ∃ X, Packing d X P
def LatticePacking ( Λ P : Set V) : Prop := (Packing d Λ P) ∧ (is_lattice Λ)
def isLatticePacking (P : Set V) : Prop := ∃ Λ, LatticePacking d Λ P
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Second attempt (4 July, 2024): A soup of unbundled predicates! Centres *still* not recoverable!

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  centers : Set (EuclideanSpace ℝ (Fin d))
  radius : ℝ
  radius_pos : 0 < radius
  centers_dist : Pairwise (radius ≤ dist · · : centers → centers → Prop)
structure PeriodicSpherePacking (d : ℕ) extends SpherePacking d where
  Λ : AddSubgroup (EuclideanSpace ℝ (Fin d))
  Λ_discrete : DiscreteTopology Λ := by infer_instance
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\begin{aligned}
a(x) := &\int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \, e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \, e^{\pi i \|x\|^2 z} \, dz \\
&- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz + 2\int_{i}^{i\infty} \phi_0(z) \, e^{\pi i \|x\|^2 z} \, dz
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a and b have three very special properties: they are **radial and Schwartz**, they are (respectively) **Fourier ± 1-eigenfunctions**, and they have **double zeroes at E_8 lattice points.**

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b(x) := &\int_{-1}^{i} \psi_S\left(\frac{-1}{z+1}\right) (z+1)^2 \; e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \psi_S\left(\frac{-1}{z-1}\right) (z-1)^2 \; e^{\pi i \|x\|^2 z} \, dz \\
&- 2 \int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz - 2 \int_{i}^{i\infty} \psi_S(z) \; e^{\pi i \|x\|^2 z} \, dz \\
g(x) := &\frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x)
\end{aligned}
$$

a and b have three very special properties: they are **radial and Schwartz**, they are (respectively) **Fourier ± 1-eigenfunctions**, and they have **double zeroes at E_8 lattice points**.

The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a very special function that is crucial to Viazovska's argument. g is defined as a linear combination of two functions a and b, which are defined in terms of quasimodular forms.

$$
\begin{aligned}
a(x) :=& \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \; e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \; e^{\pi i \|x\|^2 z}\, dz \\
&- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2\, e^{\pi i \|x\|^2 z}\, dz + 2\int_{i}^{i\infty} \phi_0(z)\; e^{\pi i \|x\|^2 z}\, dz \\
b(x) :=& \int_{-1}^{i} \psi_S\left(\frac{-1}{z+1}\right) (z+1)^2 \; e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \psi_S\left(\frac{-1}{z-1}\right) (z-1)^2 \; e^{\pi i \|x\|^2 z}\, dz \\
&- 2\int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2\, e^{\pi i \|x\|^2 z}\, dz - 2\int_{i}^{i\infty} \psi_S(z)\; e^{\pi i \|x\|^2 z}\, dz \\
g(x) :=& \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x)
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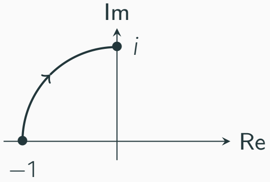
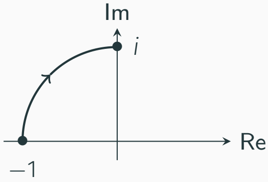
$$

a and b have three very special properties: they are **radial and Schwartz**, they are (respectively) **Fourier ± 1-eigenfunctions**, and they have **double zeroes at E_8 lattice points.** This endows g with unique, 'magic' properties.

Choosing Contours

Choosing Contours

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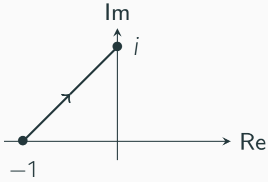
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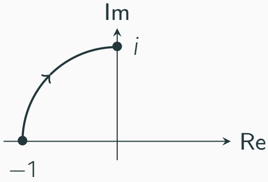

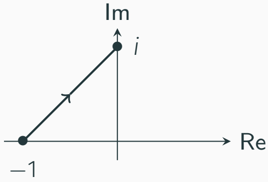

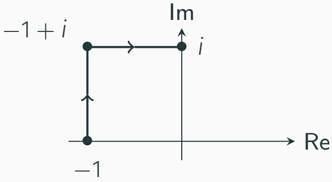

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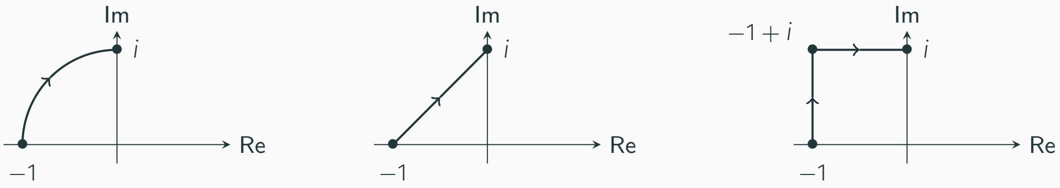


```
theorem Complex.integral_boundary_rect_eq_zero_of_differentiable_on_off_countable    source
        {E : Type u} [NormedAddCommGroup E] [NormedSpace ℂ E] (f : ℂ → E) (z w : ℂ) (s : Set ℂ)
        (hs : s.Countable) (Hc : ContinuousOn f (Set.uIcc z.re w.re ×ℂ Set.uIcc z.im w.im))
        (Hd :

         ∀ x ∈ Set.Ioo (min z.re w.re) (max z.re w.re) ×ℂ Set.Ioo (min z.im w.im) (max z.im w.im) \ s,
          DifferentiableAt ℂ f x
         )
        :
    (((∫ (x : ℝ) in z.re..w.re, f (↑x + ↑z.im * I)) - ∫ (x : ℝ) in z.re..w.re, f (↑x + ↑w.im * I)) + I •
          ∫ (y : ℝ) in z.im..w.im, f (↑w.re + ↑y * I)) - I • ∫ (y : ℝ) in z.im..w.im, f (↑z.re + ↑y * I)
          = 0
```

Cauchy-Goursat theorem for a rectangle: the integral of a complex differentiable function over the boundary of a rectangle equals zero. More precisely, if `f` is continuous on a closed rectangle and is complex differentiable at all but countably many points of the corresponding open rectangle, then its integral over the boundary of the rectangle equals zero.

Defining the Magic Function in Lean

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```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                   ⋮
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```

```
def Φ₁ : ℝ → ℂ := fun t ↦ I * Φ₁' r (z₁' t)
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```
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                    ⋮
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```
def I₁ : V → ℂ := fun x ↦ I₁' ( ‖ x ‖ ^ 2)
                ⋮
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```

```
def a' : ℝ → ℂ := fun x ↦ I₁' x + I₂' x + I₃' x + I₄' x +
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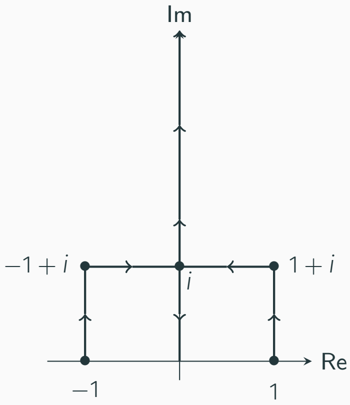
def a : V → ℂ := fun x ↦ a' ( ‖ x ‖ ^ 2)
```



Double Zeroes at Lattice Points

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$$-4\sin^2\left(\frac{\pi\left\|x\right\|^2}{2}\right)\int_0^{i\infty}\phi_0\left(\frac{-1}{z}\right)z^2\,e^{\pi i\|x\|^2 z}\,dz$$

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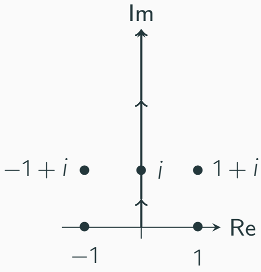

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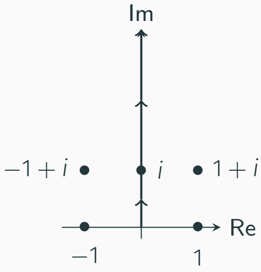
$$-4\sin^2\left(\frac{\pi\|x\|^2}{2}\right)\int_0^{i\infty}\phi_0\left(\frac{-1}{z}\right)z^2\,e^{\pi i\|x\|^2 z}\,dz=\int_{-1}^{-1+i\infty}+\int_{1}^{1+i\infty}-2\int_0^{i\infty}$$



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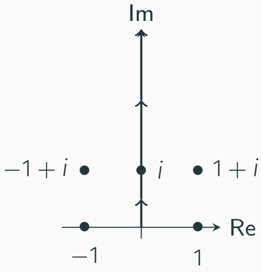

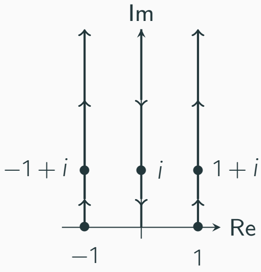

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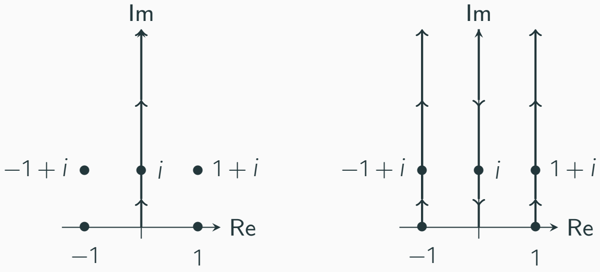
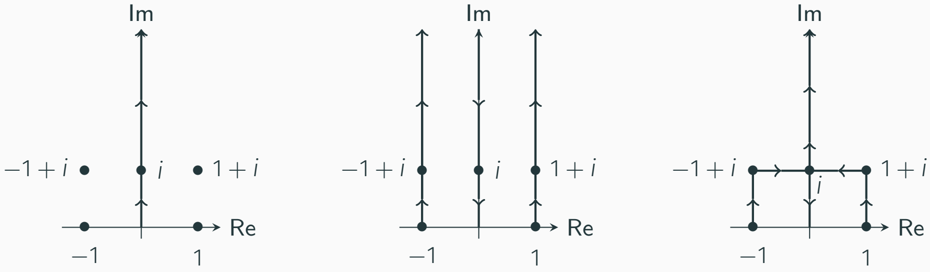


$$-4\sin^2\left(\frac{\pi\,\|x\|^2}{2}\right)\int_0^{i\infty}\phi_0\left(\frac{-1}{z}\right)z^2\,e^{\pi i\|x\|^2 z}\,dz=\int_{-1}^{-1+i\infty}+\int_{1}^{1+i\infty}-2\int_0^{i\infty}=\int_{-1}^{i}+\int_{1}^{i}-2\int_0^{i}+2\int_i^{i\infty}$$



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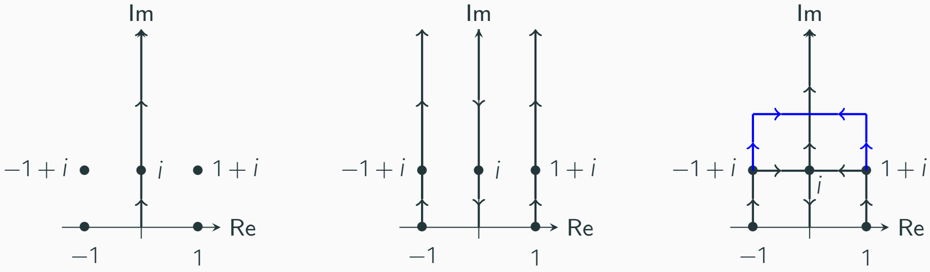

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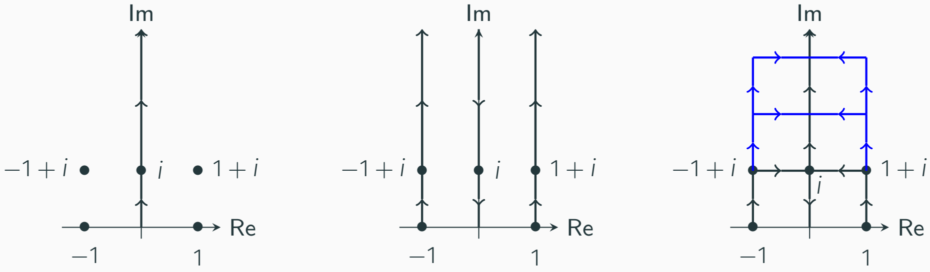

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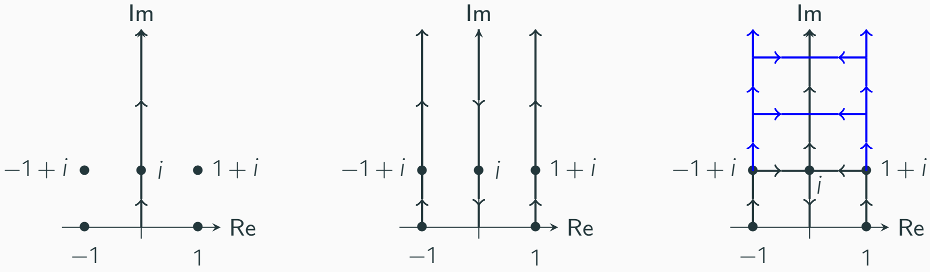

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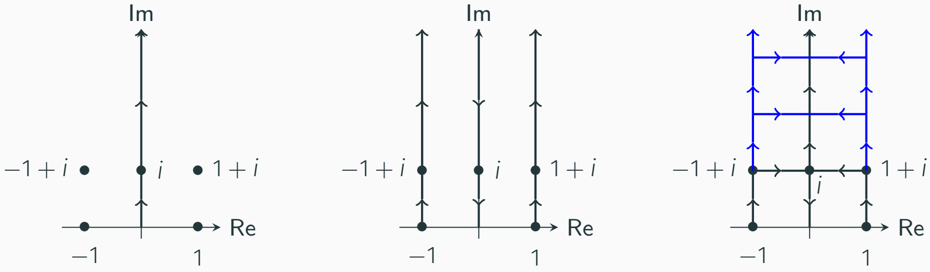

Proving Schwartzness

Fairly early on in the argument, we end up having to prove that g is a Schwartz function. This is done by showing that the constituent integrals I_k (making up a) and J_k (making up b) are Schwartz.

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It is quite simple to prove that composing with $\|\cdot\|^2$ preserves Schwartzness.

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It is quite simple to prove that composing with $\|\cdot\|^2$ preserves Schwartzness. However, I'_k and J'_k do not decay on negative inputs if defined in terms of r.

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1. One idea was to use option 1 and prove well-behavedness at 0.
2. Another idea was to use option 2, define a more general property (“Schwartz on a subset”) and study compositions preserving that property and relating it to Schwartzness.

Proving Schwartzness

Fairly early on in the argument, we end up having to prove that g is a Schwartz function. This is done by showing that the constituent integrals I_k (making up a) and J_k (making up b) are Schwartz.

It is taken for granted in the original paper that proving Schwartzness in $\|x\|$ is enough. But this is quite a nontrivial fact.

There are two ways we can define I_k and J_k:

1. We can define I'_k and J'_k in terms of r^2 and compose with $\|\cdot\|$ to obtain I_k and J_k.
2. We can define I'_k and J'_k in terms of r and compose with $\|\cdot\|^2$ to obtain I_k and J_k.

It is quite simple to prove that composing with $\|\cdot\|^2$ preserves Schwartzness. However, I'_k and J'_k do not decay on negative inputs if defined in terms of r. Here are a few solutions I considered.

1. One idea was to use option 1 and prove well-behavedness at 0.
2. Another idea was to use option 2, define a more general property ("Schwartz on a subset") and study compositions preserving that property and relating it to Schwartzness.
3. Matt Cushman proposed a third idea: to use option 2 but multiply I'_k and J'_k by a smooth transition function. I am currently implementing this.

What Truly Makes a Formalisation

More than just Confidence

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In some sense, "formalisation makes a result bigger than itself."

Modularity

Sphere Packing Theory **(**SH, GM, BM**)**

Sphere Packing Theory (SH, GM, BM)

- Definitions of sphere packings, periodic packings, density, sphere packing constant, periodic sphere packing constant
- Density formula for periodic packings
- The theory of the E_8 lattice and the associated sphere packing
- Progress on Cohn-Elkies [2, Theorem 3.1]

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Modular Forms (CB, SL)

- Fixing the definition of E_2
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- Definitions and API for a, b, g, their integrands and parametrisations
- Proving `PolyFourierCoeffBound`, a tool to analyse asymptotic behaviour
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Quasimodular Forms and Inequalities (SL)

- Serre derivatives: definitions and API, equivariance and invariance
- Serre derivatives of E_2, E_4, E_6
- Defining F and G, restriction to the imaginary axis, positivity of restrictions
- Modular linear differential equations

Metaprogramming

A `norm_num` extension for $\mathbb{C}$

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Proving `Tendsto` given atomic limits

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```
-- Two atoms, polynomial
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1)) :
   Tendsto (fun z => f z ^ 2 + f z * g z + g
    z ^ 2) atTop (nhds 1) := by tendsto_cont

-- Unused hypotheses don't interfere
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1))
   (unrelated : Tendsto f atBot (nhds 5)) :
   Tendsto (fun z => f z * g z) atTop (nhds
    0) := by tendsto_cont
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The sixteenth (**using automated reasoning**) and seventeenth (**using artificial intelligence**) have both been greatly impactful to this project, and will remain a big part of its future.

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Thank you very much! 감사합니다!

Questions**?**

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