Formalising 8-Dimensional Sphere Packing in Lean

Lean Together

Chris Birkbeck [1], Sidharth Hariharan [2], Seewoo Lee [3]

21 January, 2026

[1]The University of East Anglia

[2]Carnegie Mellon University

[3]University of California, Berkeley

Contents

1. A Bird's Eye View of the Sphere Packing Project
2. Viazovska's Magic Function
3. Dimensions of spaces of Modular forms and E_2.
4. Derivatives and Quasimodular Form Inequalities
5. Concluding Remarks

A Bird’s Eye View of the Sphere Packing Project

The sphere packing problem asks the following question:

The problem, the solution, and the way to get there

The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

The problem, the solution, and the way to get there

The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4].

The problem, the solution, and the way to get there

The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

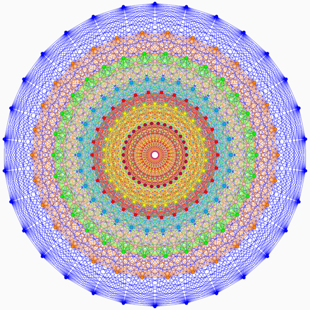
Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4]. Below, we give an illustration of a projection of the E_8 root system onto its so-called Coxeter plane [1].

The problem, the solution, and the way to get there

The sphere packing problem asks the following question:

What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

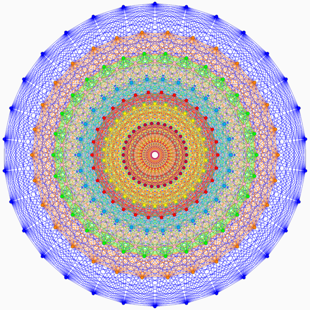
Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4]. Below, we give an illustration of a projection of the E_8 root system onto its so-called Coxeter plane [1].



The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4]. Below, we give an illustration of a projection of the E_8 root system onto its so-called Coxeter plane [1]. Nodes are vectors and edges connect nearest neighbours.



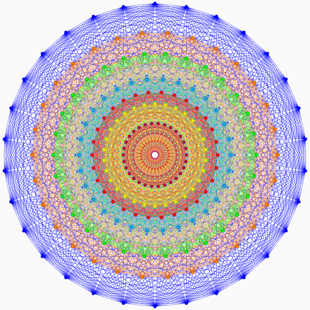
The problem, the solution, and the way to get there

The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4]. Below, we give an illustration of a projection of the E_8 root system onto its so-called Coxeter plane [1]. Nodes are vectors and edges connect nearest neighbours.

This was long suspected, and in 2003, Henry Cohn and Noam Elkies proposed a method that many believed would offer a means to a proof.

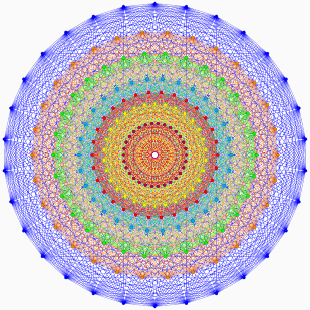


The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4]. Below, we give an illustration of a projection of the E_8 root system onto its so-called Coxeter plane [1]. Nodes are vectors and edges connect nearest neighbours.

This was long suspected, and in 2003, Henry Cohn and Noam Elkies proposed a method that many believed would offer a means to a proof. They showed that if a certain type of $\mathbb{R}^n \to \mathbb{R}$ function is constructed, then all sphere packings in $\mathbb{R}^n$ are bounded above by a certain quantity that depends on the function.



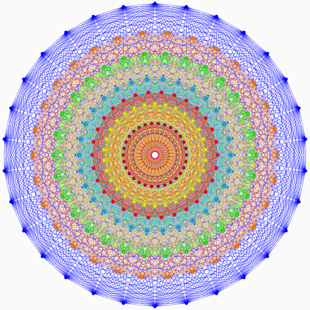
The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4]. Below, we give an illustration of a projection of the E_8 root system onto its so-called Coxeter plane [1]. Nodes are vectors and edges connect nearest neighbours.

This was long suspected, and in 2003, Henry Cohn and Noam Elkies proposed a method that many believed would offer a means to a proof. They showed that if a certain type of $\mathbb{R}^n \to \mathbb{R}$ function is constructed, then all sphere packings in $\mathbb{R}^n$ are bounded above by a certain quantity that depends on the function.

Unfortunately, constructing such functions is extremely difficult.



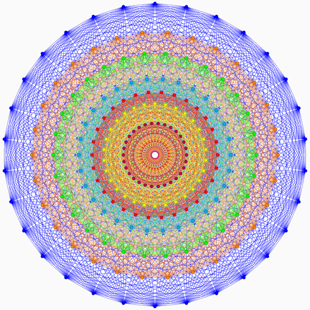
The problem, the solution, and the way to get there

The sphere packing problem asks the following question:

> What is the densest arrangement in $\mathbb{R}^n$ of non-overlapping n-spheres of the same radius?

Viazovska famously showed that the densest arrangement in $\mathbb{R}^8$ corresponds to spheres centred at points on the E_8 root lattice [4]. Below, we give an illustration of a projection of the E_8 root system onto its so-called Coxeter plane [1]. Nodes are vectors and edges connect nearest neighbours.

This was long suspected, and in 2003, Henry Cohn and Noam Elkies proposed a method that many believed would offer a means to a proof. They showed that if a certain type of $\mathbb{R}^n \to \mathbb{R}$ function is constructed, then all sphere packings in $\mathbb{R}^n$ are bounded above by a certain quantity that depends on the function.



Unfortunately, constructing such functions is extremely difficult. Viazovska's monumental breakthrough was to construct such a function using the theory of modular forms.

Project Areas

1. Sphere packing theory

Project Areas

1. Sphere packing theory
2. The Cohn-Elkies Linear Programming Bound

Project Areas

1. Sphere packing theory
2. The Cohn-Elkies Linear Programming Bound
3. Modular forms theory

Project Areas

1. Sphere packing theory
2. The Cohn-Elkies Linear Programming Bound
3. Modular forms theory
4. Constructing the magic function

Project Areas

1. Sphere packing theory
2. The Cohn-Elkies Linear Programming Bound
3. Modular forms theory
4. Constructing the magic function
5. Proving the modular form inequalities

Viazovska’s Magic Function

Defining the Magic Function

Defining the Magic Function

The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a radial Schwartz function defined as a linear combination of a Fourier $+1$-eigenfunction a and a -1-eigenfunction b.

Defining the Magic Function

The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a radial Schwartz function defined as a linear combination of a Fourier $+1$-eigenfunction a and a -1-eigenfunction b. These are defined as follows.

The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a radial Schwartz function defined as a linear combination of a Fourier $+1$-eigenfunction a and a -1-eigenfunction b. These are defined as follows.

$$a(x) := \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \; e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \; e^{\pi i \|x\|^2 z} \, dz$$
$$- 2 \int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz + 2 \int_{i}^{i\infty} \phi_0(z) \; e^{\pi i \|x\|^2 z} \, dz$$

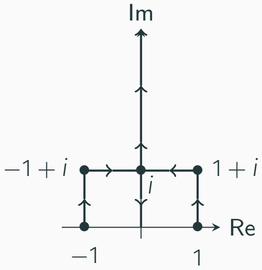
Defining the Magic Function

The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a radial Schwartz function defined as a linear combination of a Fourier $+1$-eigenfunction a and a -1-eigenfunction b. These are defined as follows.

$$
\begin{aligned}
a(x) :=& \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right) (z+1)^2 \; e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right) (z-1)^2 \; e^{\pi i \|x\|^2 z} \, dz \\
&- 2 \int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz + 2 \int_{i}^{i\infty} \phi_0(z) \; e^{\pi i \|x\|^2 z} \, dz \\
b(x) :=& \int_{-1}^{i} \psi_S\left(\frac{-1}{z+1}\right) (z+1)^2 \; e^{\pi i \|x\|^2 z} \, dz + \int_{1}^{i} \psi_S\left(\frac{-1}{z-1}\right) (z-1)^2 \; e^{\pi i \|x\|^2 z} \, dz \\
&- 2 \int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2 \, e^{\pi i \|x\|^2 z} \, dz - 2 \int_{i}^{i\infty} \psi_S(z) \; e^{\pi i \|x\|^2 z} \, dz
\end{aligned}
$$

The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a radial Schwartz function defined as a linear combination of a Fourier $+1$-eigenfunction a and a -1-eigenfunction b. These are defined as follows.

$$a(x) := \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2\, e^{\pi i \|x\|^2 z}\, dz$$
$$- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2\, e^{\pi i \|x\|^2 z}\, dz + 2\int_{i}^{i\infty} \phi_0(z)\, e^{\pi i \|x\|^2 z}\, dz$$
$$b(x) := \int_{-1}^{i} \psi_S\left(\frac{-1}{z+1}\right)(z+1)^2\, e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \psi_S\left(\frac{-1}{z-1}\right)(z-1)^2\, e^{\pi i \|x\|^2 z}\, dz$$
$$- 2\int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2\, e^{\pi i \|x\|^2 z}\, dz - 2\int_{i}^{i\infty} \psi_S(z)\, e^{\pi i \|x\|^2 z}\, dz$$



Defining the Magic Function

The magic function $g : \mathbb{R}^8 \to \mathbb{R}$ is a radial Schwartz function defined as a linear combination of a Fourier $+1$-eigenfunction a and a -1-eigenfunction b. These are defined as follows.

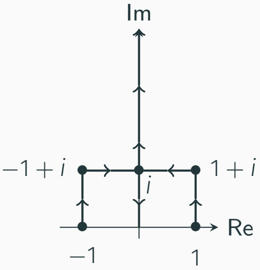
$$a(x) := \int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2\; e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2\; e^{\pi i \|x\|^2 z}\, dz$$

$$- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2\; e^{\pi i \|x\|^2 z}\, dz + 2\int_{i}^{i\infty} \phi_0(z)\; e^{\pi i \|x\|^2 z}\, dz$$

$$b(x) := \int_{-1}^{i} \psi_S\left(\frac{-1}{z+1}\right)(z+1)^2\; e^{\pi i \|x\|^2 z}\, dz + \int_{1}^{i} \psi_S\left(\frac{-1}{z-1}\right)(z-1)^2\; e^{\pi i \|x\|^2 z}\, dz$$

$$- 2\int_{0}^{i} \psi_S\left(\frac{-1}{z}\right) z^2\; e^{\pi i \|x\|^2 z}\, dz - 2\int_{i}^{i\infty} \psi_S(z)\; e^{\pi i \|x\|^2 z}\, dz$$

$$g(x) := \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x)$$



Defining the Magic Function in Lean

Defining the Magic Function in Lean

```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                          ⋮
def Φ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp ( π * I * r * (z : ℂ))
```

Defining the Magic Function in Lean

```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                 ⋮
def Φ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp ( π * I * r * (z : ℂ))
```

```
def Φ₁ : ℝ → ℂ := fun t ↦ I * Φ₁' r (z₁' t)
                ⋮
def Φ₆ : ℝ → ℂ := fun t ↦ I * Φ₆' r (z₆' t)
```

Defining the Magic Function in Lean

```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                    ⋮
def Φ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp ( π * I * r * (z : ℂ))
```

```
def Φ₁ : ℝ → ℂ := fun t ↦ I * Φ₁' r (z₁' t)
                  ⋮
def Φ₆ : ℝ → ℂ := fun t ↦ I * Φ₆' r (z₆' t)
```

```
def I₁' : ℝ → ℂ := fun x ↦ ∫ t in (0 : ℝ)..1, Φ₁ x t
                    ⋮
def I₆' : ℝ → ℂ := fun x ↦ 2 * ∫ t in Ici (1 : ℝ), Φ₆ x t
```

Defining the Magic Function in Lean

```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                  ⋮
def Φ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp ( π * I * r * (z : ℂ))
```

```
def Φ₁ : ℝ → ℂ := fun t ↦ I * Φ₁' r (z₁' t)
                 ⋮
def Φ₆ : ℝ → ℂ := fun t ↦ I * Φ₆' r (z₆' t)
```

```
def I₁' : ℝ → ℂ := fun x ↦ ∫ t in (0 : ℝ)..1, Φ₁ x t
                  ⋮
def I₆' : ℝ → ℂ := fun x ↦ 2 * ∫ t in Ici (1 : ℝ), Φ₆ x t
```

```
def I₁ : V → ℂ := fun x ↦ I₁' ( ‖ x ‖ ^ 2)
                 ⋮
def I₆ : V → ℂ := fun x ↦ I₆' ( ‖ x ‖ ^ 2)
```

Defining the Magic Function in Lean

```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                   ⋮
def Φ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp ( π * I * r * (z : ℂ))
```

```
def Φ₁ : ℝ → ℂ := fun t ↦ I * Φ₁' r (z₁' t)
                  ⋮
def Φ₆ : ℝ → ℂ := fun t ↦ I * Φ₆' r (z₆' t)
```

```
def I₁' : ℝ → ℂ := fun x ↦ ∫ t in (0 : ℝ)..1, Φ₁ x t
                   ⋮
def I₆' : ℝ → ℂ := fun x ↦ 2 * ∫ t in Ici (1 : ℝ), Φ₆ x t
```

```
def I₁ : V → ℂ := fun x ↦ I₁' ( ‖ x ‖ ^ 2)
                  ⋮
def I₆ : V → ℂ := fun x ↦ I₆' ( ‖ x ‖ ^ 2)
```

```
def a' : ℝ → ℂ := fun x ↦ I₁' x + I₂' x + I₃' x + I₄' x +
     I₅' x + I₆' x
def a : V → ℂ := fun x ↦ a' ( ‖ x ‖ ^ 2)
```

Defining the Magic Function in Lean

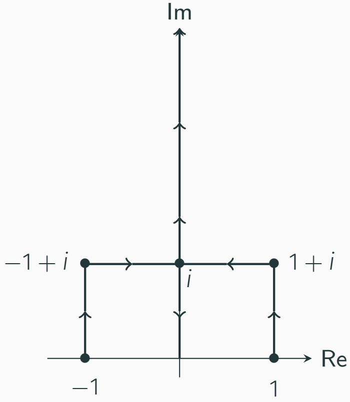
```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                    ⋮
def Φ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp ( π * I * r * (z : ℂ))
```

```
def Φ₁ : ℝ → ℂ := fun t ↦ I * Φ₁' r (z₁' t)
                   ⋮
def Φ₆ : ℝ → ℂ := fun t ↦ I * Φ₆' r (z₆' t)
```

```
def I₁' : ℝ → ℂ := fun x ↦ ∫ t in (0 : ℝ)..1, Φ₁ x t
                    ⋮
def I₆' : ℝ → ℂ := fun x ↦ 2 * ∫ t in Ici (1 : ℝ), Φ₆ x t
```

```
def I₁ : V → ℂ := fun x ↦ I₁' ( ‖ x ‖ ^ 2)
                   ⋮
def I₆ : V → ℂ := fun x ↦ I₆' ( ‖ x ‖ ^ 2)
```

```
def a' : ℝ → ℂ := fun x ↦ I₁' x + I₂' x + I₃' x + I₄' x +
     I₅' x + I₆' x
def a : V → ℂ := fun x ↦ a' ( ‖ x ‖ ^ 2)
```



Some Active and Remaining Goals (for this portion of the project)

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that $\Phi_1 \cdots \Phi_6$ are integrable over the appropriate intervals

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that `Φ₁⋯Φ₆` are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of $\mathtt{I_1'}$…$\mathtt{I_6'}$: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that $\mathbf{\Phi_1}$…$\mathbf{\Phi_6}$ are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that a has double zeroes at lattice points

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of $I_1' \cdots I_6'$: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that $\Phi_1 \cdots \Phi_6$ are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that a has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that `Φ₁⋯Φ₆` are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that *a* has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude
5. Bounding the integrals that make up *b*

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that `Φ₁⋯Φ₆` are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that a has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude
5. Bounding the integrals that make up b: I will do this once we've completed everything involving a

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of $\mathtt{I_1'}\cdots\mathtt{I_6'}$: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that $\Phi_1\cdots\Phi_6$ are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that a has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude
5. Bounding the integrals that make up b: I will do this once we've completed everything involving a
6. … and many more

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that $\Phi_1 \cdots \Phi_6$ are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that a has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude
5. Bounding the integrals that make up b: I will do this once we've completed everything involving a
6. … and many more

Interested in getting involved?

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that `Φ₁⋯Φ₆` are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that a has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude
5. Bounding the integrals that make up b: I will do this once we've completed everything involving a
6. … and many more

Interested in getting involved?

Come to our next packathon!

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that `Φ₁⋯Φ₆` are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that a has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude
5. Bounding the integrals that make up b: I will do this once we've completed everything involving a
6. … and many more

Interested in getting involved?

Come to our next packathon!

Tuesdays 10:30 am Eastern Time.

Some Active and Remaining Goals (for this portion of the project)

1. Proving one-dimensional Schwartzness of `I₁'⋯I₆'`: Done modulo minor smoothness results
2. Moving from one-dimensional to eight-dimensional Schwartzness: Actively being worked on by Matthew Cushman and myself (using Aristotle)
3. Proving that `Φ₁⋯Φ₆` are integrable over the appropriate intervals: Actively being worked on by Cameron Freer (using Claude), Tito Sacchi and myself
4. Proving that *a* has double zeroes at lattice points: Mostly completed by Tito Sacchi and ItaLean 2025 participants; some small TODOs remaining, being rebased with another PR by Cameron + Claude
5. Bounding the integrals that make up *b*: I will do this once we've completed everything involving *a*
6. … and many more

Interested in getting involved?

Come to our next packathon!

Tuesdays 10:30 am Eastern Time.

See Zulip for more

Dimensions of spaces of Modular forms and E_2.

Main result

- Modular forms and the key to us constructing our magic function. They are holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ such that for some integer k (called the weight) and subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ we have
$$f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k} = f(z)$$
for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$
- We define Eisenstein series E_k which are the ”first” examples of modular forms you see.
- We need to prove identities between them, many of which require arguments of the form ”these are two modular forms in a 1-dimensional space, so up to some constant they are equal”.
- This requires us to know some dimensions for spaces of modular forms (in level one).
- There is WIP by David Loeffler to prove that these spaces (of any level) are finite dimensional.
- I am just going to briefly describe how we did this and how this is turning into mathlib PRs.

Dimension formula (level one)

For even weights $k \geq 3$, the space of level-one modular forms

$$M_k(\mathrm{SL}_2(\mathbb{Z}))$$

has dimension

$$\dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \left\lfloor \dfrac{k}{12} \right\rfloor & \text{if } k \equiv 2 \pmod{12}, \\ \left\lfloor \dfrac{k}{12} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

Examples: $\dim M_4 = 1,\ \dim M_6 = 1,\ \dim M_{12} = 2,\ \dim M_{14} = 1.$

Dimension formula (level one)

For even weights $k \geq 3$, the space of level-one modular forms

$$M_k(\mathrm{SL}_2(\mathbb{Z}))$$

has dimension

$$\dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor & \text{if } k \equiv 2 \pmod{12}, \\ \left\lfloor \frac{k}{12} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

Examples: $\dim M_4 = 1,\ \dim M_6 = 1,\ \dim M_{12} = 2,\ \dim M_{14} = 1.$

```
lemma ModularForm.dimension_level_one (k : ℕ) (hk : 3 ≤ (k : ℤ)) (hk2 : Even k) :
    Module.rank ℂ (ModularForm (CongruenceSubgroup.Gamma 1) k) = if 12 ∣ (k : ℤ) - 2 then
    Nat.floor ((k : ℚ)/ 12) else Nat.floor ((k : ℚ) / 12) + 1 := by
  exact dim_modforms_lvl_one k hk hk2
```

Problem

- Normally this would be proven using by doing some "tricky" contour integrals, but we can't yet do these in Lean (can be done in Isabelle).
- Instead we first proved that $\dim M_0 = 1$ and $dim M_{k<0} = 0$ and then use a special NOT-modular-form

$$E_2(z) = \sum_m \sum_n \frac{1}{(mz+n)^2}.$$

- E_2 allows us to construct a special (non vanishing) modular form called

$$\Delta = q \prod_n (1 - q^n)^{24}$$

(with $q = e^{2\pi i z}$) as well as being needed later for the Serre derivatives of modular forms.

E_2

- The reason E_2 is not quite a modular form, is that $E_2|\gamma = E_2 - C_2(\gamma)$ where $C_2(\gamma)$ is some correction factor. One then checks that the logarithmic derivative (e.g. f'/f) of $\Delta^{1/24}$ is E_2 which lets us prove that Δ is a modular form.
- Since $\sum_m \sum_n \frac{1}{(mz+n)^2}$ is not absolutely convergent the order matters and the proof of what we want boils down to proving that :

```
lemma tsum_symmetricIco_tsum_sub_eq :
    ∑'[symmetricIco ℤ] n : ℤ, ∑' m : ℤ, (1 / ((m : ℂ) * z + n) - 1 / (m * z + n + 1)) =
    -2 * π * I / z := by ...


lemma tsum_tsum_symmetricIco_sub_eq :
    ∑' m : ℤ, ∑'[symmetricIco ℤ] n : ℤ, (1 / ((m : ℂ) * z + n) - 1 / (m * z + n + 1)) = 0 :=
     by ...
```

Unconditional sum API

- Together with David Loeffler, we developed an unconditional sum API (based on a suggestion of Mario Carneiro).
- ```
  -- changed from this
  def HasProd (f : β → α) (a : α) : Prop :=
    Tendsto (fun s : Finset β ↦ ∏ b ∈ s, f b) atTop (nhds a)

  -- to this
  def HasProd (f : β → α) (a : α) (L := unconditional β) : Prop :=
    Tendsto (fun s : Finset β ↦ ∏ b ∈ s, f b) L.filter (nhds a)
  ```
- The use of an optional parameters '(L := unconditional β)' was really handy!
- E_2 has been defined in mathlib now, and its transformation properties is an open PR.
- The facts about Δ are PRs in progress (we have $\Delta^{1/24}$ in mathlib already). From this the dimension formulas will follow.
  ```

Derivatives and Quasimodular Form Inequalities

The linear constrains $g(x) \leq 0$ and $\widehat{g}(x) \geq 0$ reduce to the inequalities

$$F(it) + \frac{18}{\pi^2} G(it) > 0, \quad F(it) - \frac{18}{\pi^2} G(it) < 0$$

where

$$F(z) = (E_2(z)E_4(z) - E_6(z))^2, \quad G(z) = H_2(z)^3(2H_2(z)^2 + 5H_2(z)H_4(z) + 5H_4(z)^2)$$

(Here $H_2 = \Theta_2^4$ and $H_4 = \Theta_2^4$ are 4-th powers of Jacobi theta functions.) There are *three* (informal) proofs of the inequalities:

1. Viazovska [4]: Bounds of Fourier coefficients of modular forms + interval arithmetic
2. Romik [3]: New modular form identities + explicit values at $z = i$
3. Lee [2]: (Serre) derivatives of quasimodular forms

We formalize the 3rd proof which is the easiest to formalize.

Some (informal) math

The idea of [2] for the second inequality is[1]

1. $\lim_{t \to 0^+} F(it)/G(it) = 18/\pi^2$
2. $F(it)/G(it)$ is monotone decreasing in $t > 0$

The limit can be computed using transformation laws of F and G (which requires that of E_2). Monotonicity follows from auxiliary results on (Serre) derivatives and bunch of identities between quasimodular forms, which can be proved using Ramanujan's identities.

[1] the first one trivially (informally) follows from $F(it) > 0$ and $G(it) > 0$

Derivative and Serre derivative

```
noncomputable def D (F : ℍ → ℂ) : ℍ → ℂ :=
  fun (z : ℍ) => (2 * π * I)⁻¹ * ((deriv (F ∘ ofComplex)) z)

noncomputable def serre_D (k : ℂ) : (ℍ → ℂ) → (ℍ → ℂ) :=
  fun (F : ℍ → ℂ) => (fun z => D F z - k * 12⁻¹ * E₂ z * F z)
```

Derivative D and serre derivative $\partial_k = D - \frac{k}{12}E_2$ are defined as above. Main theorems are

- `add`, `sub`, `mul`, ...
- q-expanion: $D(\sum_n a_n q^n) = \sum_n n a_n q^n$ for $q = e^{2\pi i z}$.
- (In progress) Ramanujan's identities: $\partial_1 E_2 = -\frac{1}{12}E_4, \partial_4 E_4 = -\frac{1}{3}E_6, \partial_6 E_6 = -\frac{1}{2}E_4^2$.
- (In progress) For theta functions, $\partial_2 H_2 = \frac{1}{6}(H_2^2 + 2H_2H_4)$ and $\partial_2 H_4 = -\frac{1}{6}(2H_2H_4 + H_4^2)$

Last two use dimension formula mentioned above.

Restriction to the positive imaginary axis

```
noncomputable def ResToImagAxis (F : ℍ → ℂ) : ℝ → ℂ :=
  fun t => if ht : 0 < t then F ⟨(I * t), by simp [ht]⟩ else 0

noncomputable def ResToImagAxis.Real (F : ℍ → ℂ) : Prop :=
  ∀ t : ℝ, 0 < t → (F.resToImagAxis t).im = 0

noncomputable def ResToImagAxis.Pos (F : ℍ → ℂ) : Prop :=
  ResToImagAxis.Real F ∧ ∀ t : ℝ, 0 < t → 0 < (F.resToImagAxis t).re
```

We made API `ResToImagAxis` and prove theorems on realness (`ResToImagAxis.Real`) and positivity (`ResToImagAxis.Pos`) of quasimodular forms.

- `add`, `sub`, `mul`, `pow`, `const_mul`, ...
- Realness of E_2, E_4, E_6, H_2, H_4
- Positivity of $-E_2', E_4', H_2, H_4$
- $DF(it) > 0$ and $F(it) > 0$ for sufficieintly large $t \Rightarrow F(it) > 0$
- (In progress) $\partial_k F(it) > 0$ and $F(it) > 0$ for sufficiently large $t \Rightarrow F(it) > 0$

Register `ResToImagAxis` to `fun_prop`

```
noncomputable def ResToImagAxis (F : ℍ → ℂ) : ℝ → ℂ :=
  fun t => if ht : 0 < t then F ⟨(I * t), by simp [ht]⟩ else 0

@[fun_prop]
noncomputable def ResToImagAxis.Real (F : ℍ → ℂ) : Prop :=
  ∀ t : ℝ, 0 < t → (F.resToImagAxis t).im = 0

@[fun_prop]
noncomputable def ResToImagAxis.Pos (F : ℍ → ℂ) : Prop :=
  ResToImagAxis.Real F ∧ ∀ t : ℝ, 0 < t → 0 < (F.resToImagAxis t).re
```

We can even register `ResToImagAxis` to the `fun_prop` tactic. We register it and related theorems, along with realness/positivity of "basic" quasimodular forms (E_2, E_4, E_6, H_2, H_4) to `fun_prop`, so that we can prove realness/positivity of quasimodular forms that are polynomials in the above forms `by fun_prop`.

Register `ResToImagAxis` to `fun_prop`

For example, we can go from

```
-- F = (E₂ * E₄.toFun - E₆.toFun) ^ 2
theorem F_imag_axis_real : ResToImagAxis.Real F := by
  unfold F
  have hProd : ResToImagAxis.Real (E₂ * E₄.toFun) :=
    ResToImagAxis.Real.mul E₂_imag_axis_real E₄_imag_axis_real
  have hNeg : ResToImagAxis.Real ((-1 : ℝ) • E₆.toFun) :=
    ResToImagAxis.Real.smul E₆_imag_axis_real
  have hSub : ResToImagAxis.Real (E₂ * E₄.toFun - E₆.toFun) := by
    have hEq : E₂ * E₄.toFun - E₆.toFun = E₂ * E₄.toFun + (-1 : ℝ) • E₆.toFun := by
      ext z
      simp [sub_eq_add_neg]
    simpa [hEq] using ResToImagAxis.Real.add hProd hNeg
  simpa [pow_two] using ResToImagAxis.Real.mul hSub hSub
```

Register `ResToImagAxis` to `fun_prop`

For example, we can go from

```
-- F = (E₂ * E₄.toFun - E₆.toFun) ^ 2
theorem F_imag_axis_real : ResToImagAxis.Real F := by
  unfold F
  have hProd : ResToImagAxis.Real (E₂ * E₄.toFun) :=
    ResToImagAxis.Real.mul E₂_imag_axis_real E₄_imag_axis_real
  have hNeg : ResToImagAxis.Real ((-1 : ℝ) • E₆.toFun) :=
    ResToImagAxis.Real.smul E₆_imag_axis_real
  have hSub : ResToImagAxis.Real (E₂ * E₄.toFun - E₆.toFun) := by
    have hEq : E₂ * E₄.toFun - E₆.toFun = E₂ * E₄.toFun + (-1 : ℝ) • E₆.toFun := by
      ext z
      simp [sub_eq_add_neg]
    simpa [hEq] using ResToImagAxis.Real.add hProd hNeg
  simpa [pow_two] using ResToImagAxis.Real.mul hSub hSub
```

to

```
theorem F_imag_axis_real : ResToImagAxis.Real F := by unfold F; fun_prop
```

Register `MDifferentiable` to `fun_prop`

We can do similar for `MDifferentiable` (holomorphicity). We register it and related theorems, along with holomorphicity of "basic" quasimodular forms (E_2, E_4, E_6, H_2, H_4) to `fun_prop`, so that we can prove holomorphicity of quasimodular forms that are polynomials in the above forms `by fun_prop`. '

```
attribute [fun_prop] MDifferentiable

attribute [fun_prop]
  MDifferentiable.add
  MDifferentiable.sub
  MDifferentiable.neg
  MDifferentiable.mul
  MDifferentiable.pow
  MDifferentiable.const_smul
  mdifferentiable_const
  E₄_MDifferentiable
  E₆_MDifferentiable
```

Register `MDifferentiable` to `fun_prop`

For example, we can go from

```
theorem F_aux : ... := by  -- some auxiliary identity
  ...
  -- Holomorphicity of the terms
  · exact E₂_holo'
  · exact E₄.holo'
  · exact MDifferentiable.mul E₂_holo' E₄.holo'
  · exact E₆.holo'
  · exact MDifferentiable.sub (MDifferentiable.mul E₂_holo' E₄.holo') E₆.holo'
```

Register `MDifferentiable` to `fun_prop`

For example, we can go from

```
theorem F_aux : ... := by  -- some auxiliary identity
  ...
  -- Holomorphicity of the terms
  · exact E₂_holo'
  · exact E₄.holo'
  · exact MDifferentiable.mul E₂_holo' E₄.holo'
  · exact E₆.holo'
  · exact MDifferentiable.sub (MDifferentiable.mul E₂_holo' E₄.holo') E₆.holo'
```

to

```
theorem F_aux : ... := by  -- some auxiliary identity
  ...
  -- Holomorphicity of the terms
  repeat fun_prop
```

Remaining Goals

- More modular form identities and positivity results (already done in PRs by Cameron + Claude)
- Limit computation $\lim_{t\to 0^+} F(it)/G(it) = 18/\pi^2$, and finish proof of the inequalities
- Connect them with the actual linear constraints of the magic function ($g \le 0$, $\widehat{g} \ge 0$)
- Upstream (Serre) derivative and $\partial_k(M_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq M_{k+2}(\mathrm{SL}_2(\mathbb{Z}))$ to `mathlib`
- Upstream Jacobi identity $\Theta_2^4 + \Theta_4^4 = \Theta_3^4$ to `mathlib`

Concluding Remarks

Getting Involved in the Sphere Packing Project

Formalising Sphere Packing in Lean

by Chris Birkbeck, Sidharth Hariharan, Gareth Ma, Bhavik Mehta, Seewoo Lee, Maryna Viazovska

Blueprint (web) | Blueprint (pdf) | Documentation | GitHub

Formalising Sphere Packing in Lean

In 2022, Maryna Viazovska was awarded the Fields Medal for solving the sphere packing problem in dimension 8. This project formalises this result in the Lean Theorem Prover following her original paper and followup calculations by Lee. It is an online, open-source collaboration currently being led by Christopher Birkbeck, Sidharth Hariharan, Bhavik Mehta and Seewoo Lee. If you'd like to contribute, you may find the following links useful!

- Zulip chat for coordination
- Blueprint
- Blueprint as pdf
- Dependency graph
- Doc pages for this repository
- Project dashboard
- Viazovska's original paper

SCAN TO VISIT OUR WEBSITE!

Formalising Sphere Packing in Lean

by Chris Birkbeck, Sidharth Hariharan, Gareth Ma, Bhavik Mehta, Seewoo Lee, Maryna Viazovska

Blueprint (web) | Blueprint (pdf) | Documentation | GitHub

Formalising Sphere Packing in Lean

In 2022, Maryna Viazovska was awarded the Fields Medal for solving the sphere packing problem in dimension 8. This project formalises this result in the Lean Theorem Prover following her original paper and followup calculations by Lee. It is an online, open-source collaboration currently being led by Christopher Birkbeck, Sidharth Hariharan, Bhavik Mehta and Seewoo Lee. If you'd like to contribute, you may find the following links useful!

- Zulip chat for coordination
- Blueprint
- Blueprint as pdf
- Dependency graph
- Doc pages for this repository
- Project dashboard
- Viazovska's original paper

SCAN TO VISIT OUR WEBSITE!

Don't forget to come to our next packathon: Tuesday 27 January at 10:30 am Eastern Time, IRL @ Hoskinson or on Zoom at `cmu.zoom.us/my/sidharth.hariharan`. For updates and discussion, see Zulip.

Getting Involved in the Sphere Packing Project

Don't forget to come to our next packathon: Tuesday 27 January at 10:30 am Eastern Time, IRL @ Hoskinson or on Zoom at `cmu.zoom.us/my/sidharth.hariharan`. For updates and discussion, see Zulip.

All links can be found on the project website: `https://thefundamentaltheor3m.github.io/Sphere-Packing-Lean`

Acknowledgements

Acknowledgements

- Maryna Viazovska

Acknowledgements

- Maryna Viazovska
- Kevin Buzzard

Acknowledgements

- Maryna Viazovska
- Kevin Buzzard
- Bhavik Mehta

Acknowledgements

- Maryna Viazovska
- Kevin Buzzard
- Bhavik Mehta
- Jeremy Avigad, Hoskinson Center for Formal Mathematics, Institute for Computer-Aided Reasoning in Mathematics

- Maryna Viazovska
- Kevin Buzzard
- Bhavik Mehta
- Jeremy Avigad, Hoskinson Center for Formal Mathematics, Institute for Computer-Aided Reasoning in Mathematics
- David Renshaw

Acknowledgements

- Maryna Viazovska
- Kevin Buzzard
- Bhavik Mehta
- Jeremy Avigad, Hoskinson Center for Formal Mathematics, Institute for Computer-Aided Reasoning in Mathematics
- David Renshaw
- Utensil Song and Pietro Monticone

Acknowledgements

- Maryna Viazovska
- Kevin Buzzard
- Bhavik Mehta
- Jeremy Avigad, Hoskinson Center for Formal Mathematics, Institute for Computer-Aided Reasoning in Mathematics
- David Renshaw
- Utensil Song and Pietro Monticone
- All our contributors! (Human and non-human alike)

 and

Questions?

References i

[1] T. F. Görbe.
Exceptionally Beautiful Symmetries.
`https://tamasgorbe.com/symmetry`.

[2] S. Lee.
Algebraic proof of modular form inequalities for optimal sphere packings.
arXiv preprint arXiv:2406.14659, 2024.

[3] D. Romik.
On viazovska's modular form inequalities.
Proceedings of the National Academy of Sciences, 120(43):e2304891120, 2023.

[4] M. S. Viazovska.
The sphere packing problem in dimension 8.
Annals of Mathematics, 185(3):991–1015, 2017.