Carnegie Mellon University

A Lean Formalisation of Sphere Packing in Dimension 8

Swiss Mathematical Society Spring Meeting: Formalisation and Proof Assistants

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based on joint work with Chris Birkbeck, Seewoo Lee, Gareth Ma, Bhavik Mehta and Maryna Viazovska

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Project Blueprint

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`https://thefundamentaltheor3m.github.io/Sphere-Packing-Lean/blueprint`

Defining Sphere Packings in Lean

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structure SpherePacking (d : ℕ) where
  centers : Set (EuclideanSpace ℝ (Fin d))
  separation : ℝ
  separation_pos : 0 < separation := by positivity
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```
structure PeriodicSpherePacking (d : ℕ) extends SpherePacking d where
  lattice : Submodule ℤ (EuclideanSpace ℝ (Fin d))
  lattice_action : ∀ ⦃ x y ⦄, x ∈ lattice → y ∈ centers → x + y ∈ centers
  lattice_discrete : DiscreteTopology lattice := by infer_instance
  lattice_isZLattice : IsZLattice ℝ lattice := by infer_instance
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```

```
abbrev SpherePacking.balls (S : SpherePacking d) : Set (EuclideanSpace ℝ (Fin d)) :=
  ⋃ x : S.centers, ball (x : EuclideanSpace ℝ (Fin d)) (S.separation / 2)
```

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noncomputable def SpherePacking.density (S : SpherePacking d) : ℝ ≥ 0 ∞ :=
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```
noncomputable def E8Packing : PeriodicSpherePacking 8 where
  separation := √2
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```
theorem SpherePacking.MainTheorem : SpherePackingConstant 8 = E8Packing.density :=
  sorry
```

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Sphere Packing Theory

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Sphere Packing Theory

- Definitions of sphere packings, periodic packings, density, sphere packing constant, periodic sphere packing constant
- Density formula for periodic packings
- The theory of the E_8 lattice and the associated sphere packing
- Progress on Cohn-Elkies [2, Theorem 3.1]

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Modular Forms

- Fixing the definition of E_2
- q-expansions of the Eisenstein Series
- H-functions (4th powers of Jacobi Θ-functions) as modular forms
- Definition/API for Δ, relation to η
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- Definitions and API for a, b, g, their integrands and parametrisations
- Proving `PolyFourierCoeffBound`, a tool to analyse asymptotic behaviour
- Systematically bounding the integrands of a
- Proving slash action relations for ψ_I, ψ_S, ψ_T
- Cauchy-Goursat for unbounded rectangles

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Quasimodular Forms and Inequalities

- Serre derivatives: definitions and API, equivariance and invariance
- Serre derivatives of E_2, E_4, E_6
- Defining F and G, restriction to the imaginary axis, positivity of restrictions
- Modular linear differential equations

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- Simplifies expressions in $\mathbb{C}$ by splitting to real and imaginary parts, running `norm_num` on each part, and recombining them

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example : (1 + 3.5 + I) * (1 + I) = 7 / 2 +
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- Proves `Tendsto (fun z => expr(f₁ z, ..., f ₙ z)) l (nhds c)` where `Tendsto f ᵢ l (nhds a ᵢ)` are known and the expression is continuous

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```
-- Two atoms, polynomial
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1)) :
    Tendsto (fun z => f z ^ 2 + f z * g z + g
    z ^ 2) atTop (nhds 1) := by tendsto_cont

-- Unused hypotheses don't interfere
example (h₁ : Tendsto f atTop (nhds 0)) (h₂ :
    Tendsto g atTop (nhds 1))
    (unrelated : Tendsto f atBot (nhds 5)) :
    Tendsto (fun z => f z * g z) atTop (nhds
    0) := by tendsto_cont
```

Active Goals (as of our Lean Together talk on 21 January)

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- Schwartzness: one-dimensional and eight-dimensional
- Proving that $\Phi_1 \cdots \Phi_6$ are integrable
- Proving that a has double zeroes at lattice points
- Bounding the integrals that make up b

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Modular Forms, Quasimodular Forms, and Inequalities

- More modular form identities and positivity results
- Limit computation $\lim_{t \to 0^+} F(it)/G(it) = 18/\pi^2$, and finish proof of the inequalities
- Dimension formula in level 1
- Upstreaming the definition/facts about the Serre Derivative, the proof of the Jacobi identity $\Theta_2^4 + \Theta_4^4 = \Theta_3^4$, and facts about E_2 and Δ to `mathlib`.

Contour Integration

Defining the Magic Function in Lean

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```
def Φ₁' : ℂ → ℂ := fun z ↦ φ₀'' (-1 / (z + 1)) * (z + 1) ^ 2 * cexp ( π * I * r * (z : ℂ))
                   ⋮
def Φ₆' : ℂ → ℂ := fun z ↦ φ₀'' (z) * cexp ( π * I * r * (z : ℂ))
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def I₁' : ℝ → ℂ := fun x ↦ ∫ t in (0 : ℝ)..1, Φ₁ x t
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def I₆' : ℝ → ℂ := fun x ↦ 2 * ∫ t in Ici (1 : ℝ), Φ₆ x t
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```
def I₁ : V → ℂ := fun x ↦ I₁' ( ‖ x ‖ ^ 2)
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```
def a' : ℝ → ℂ := fun x ↦ I₁' x + I₂' x + I₃' x + I₄' x +
    I₅' x + I₆' x
def a : V → ℂ := fun x ↦ a' ( ‖ x ‖ ^ 2)
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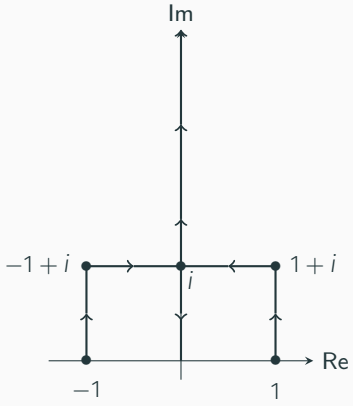
```

```
def I₁ : V → ℂ := fun x ↦ I₁' ( ‖ x ‖ ^ 2)
                  ⋮
def I₆ : V → ℂ := fun x ↦ I₆' ( ‖ x ‖ ^ 2)
```

```
def a' : ℝ → ℂ := fun x ↦ I₁' x + I₂' x + I₃' x + I₄' x +
     I₅' x + I₆' x
def a : V → ℂ := fun x ↦ a' ( ‖ x ‖ ^ 2)
```



Why use rectangular contours?

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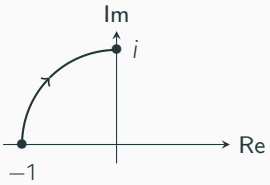
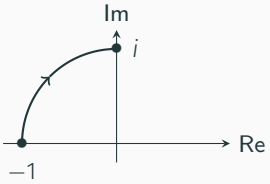
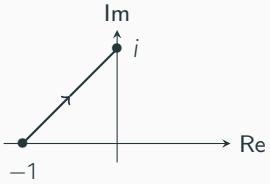


Why use rectangular contours?





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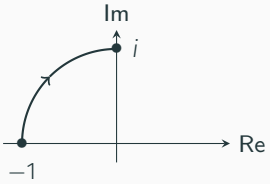

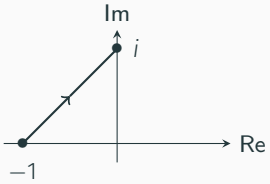

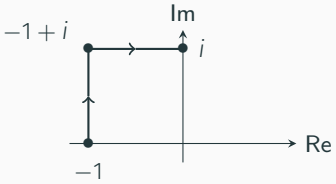

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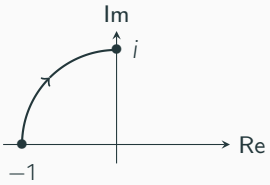

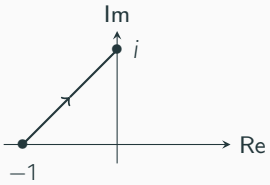

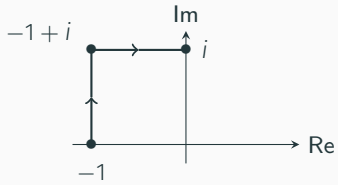


```
theorem Complex.integral_boundary_rect_eq_zero_of_differentiable_on_off_countable
        {E : Type u} [NormedAddCommGroup E] [NormedSpace ℂ E] (f : ℂ → E) (z w : ℂ) (s : Set ℂ)
        (hs : s.Countable) (Hc : ContinuousOn f (Set.uIcc z.re w.re ×ℂ Set.uIcc z.im w.im))
        (Hd :

         ∀ x ∈ Set.Ioo (min z.re w.re) (max z.re w.re) ×ℂ Set.Ioo (min z.im w.im) (max z.im w.im) \ s,
          DifferentiableAt ℂ f x
         )
        :
    (((∫ (x : ℝ) in z.re..w.re, f (↑x + ↑z.im * I)) - ∫ (x : ℝ) in z.re..w.re, f (↑x + ↑w.im * I)) + I •
          ∫ (y : ℝ) in z.im..w.im, f (↑w.re + ↑y * I)) - I • ∫ (y : ℝ) in z.im..w.im, f (↑z.re + ↑y * I)
          = 0
```

source

Cauchy-Goursat theorem for a rectangle: the integral of a complex differentiable function over the boundary of a rectangle equals zero. More precisely, if `f` is continuous on a closed rectangle and is complex differentiable at all but countably many points of the corresponding open rectangle, then its integral over the boundary of the rectangle equals zero.

Double Zeroes at Lattice Points

Double Zeroes at Lattice Points

$$-4\sin^2\left(\frac{\pi\,\|x\|^2}{2}\right)\int_0^{i\infty}\phi_0\left(\frac{-1}{z}\right)z^2\,e^{\pi i\|x\|^2 z}\,dz$$

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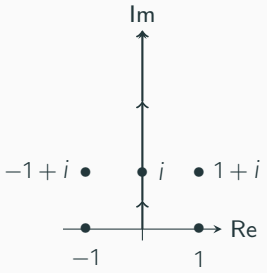
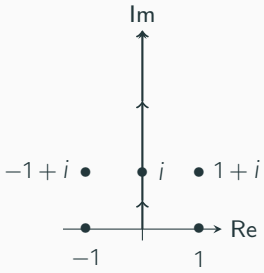


Double Zeroes at Lattice Points

$$-4\sin^2\left(\frac{\pi\,\|x\|^2}{2}\right)\int_0^{i\infty}\phi_0\left(\frac{-1}{z}\right)z^2\,e^{\pi i\|x\|^2 z}\,dz=\int_{-1}^{-1+i\infty}+\int_1^{1+i\infty}-2\int_0^{i\infty}$$



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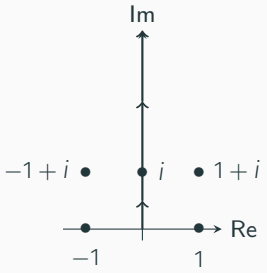

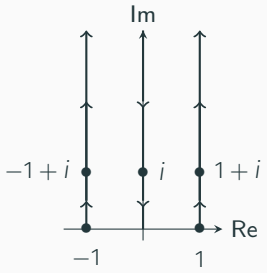

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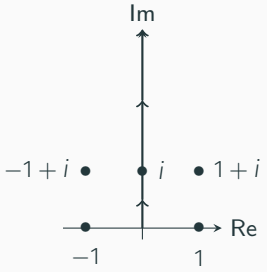

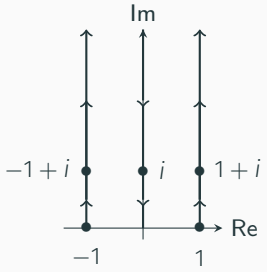

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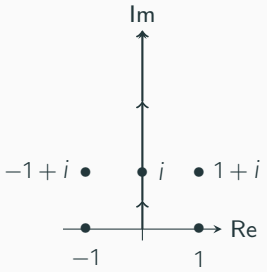

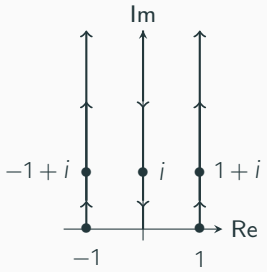

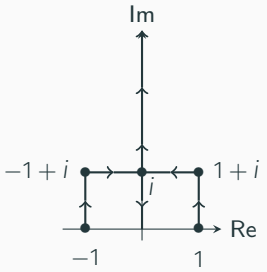

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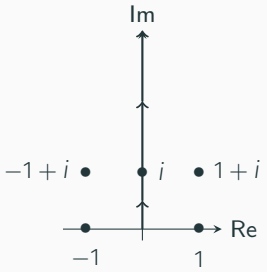

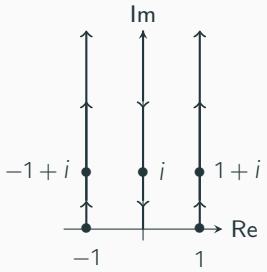

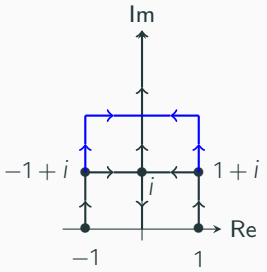

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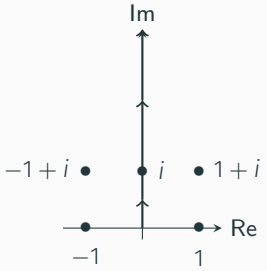

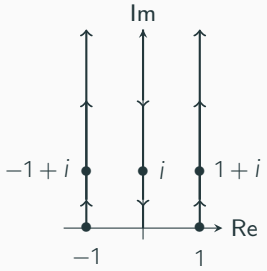

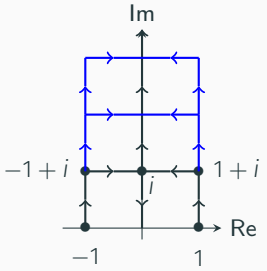

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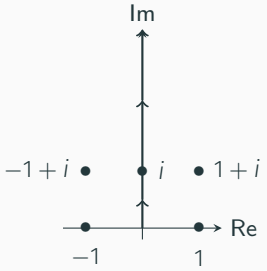

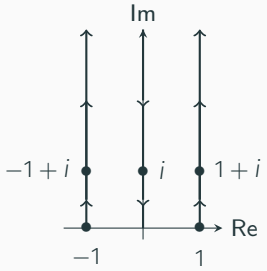

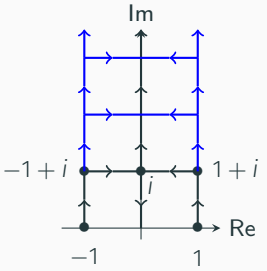

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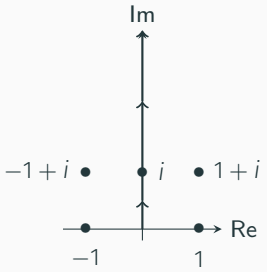
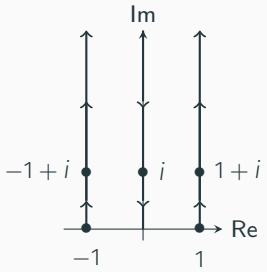
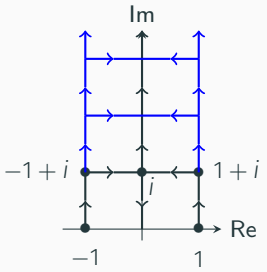


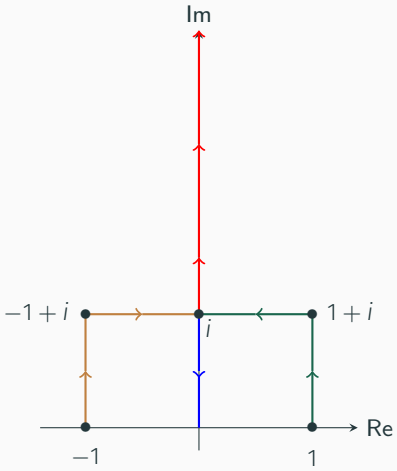




The Fourier Eigenfunction Property

The Fourier Eigenfunction Property

Before taking the Fourier transform:



The Fourier Eigenfunction Property

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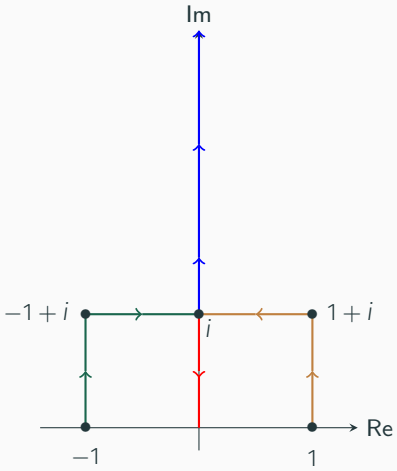

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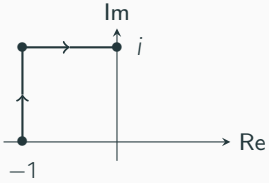
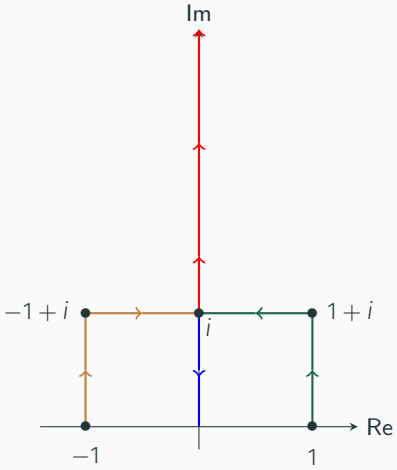
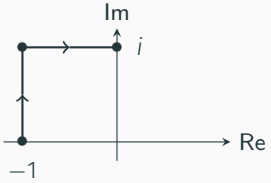
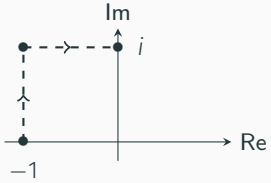
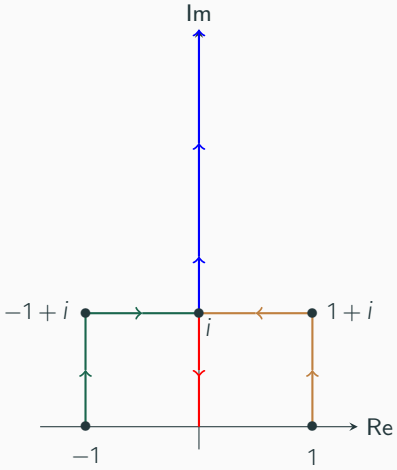




After taking the Fourier transform:







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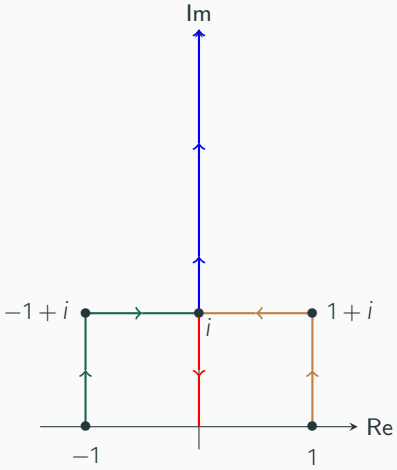

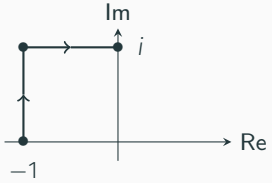

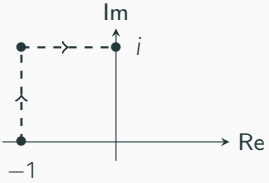

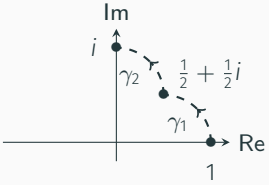

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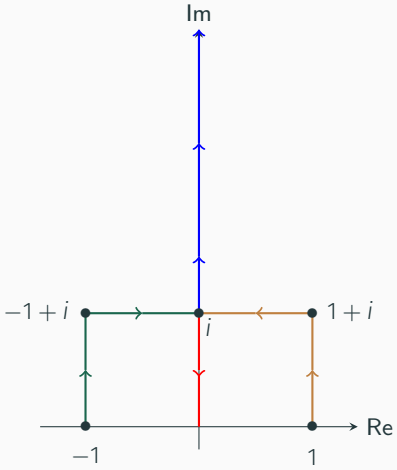

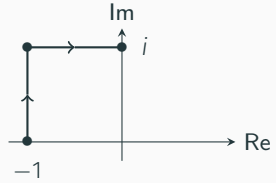

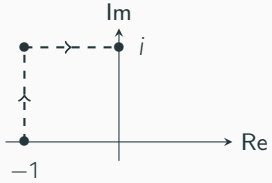

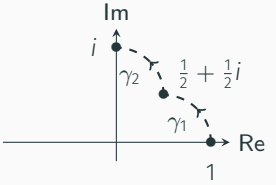

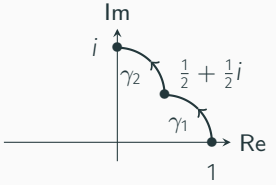

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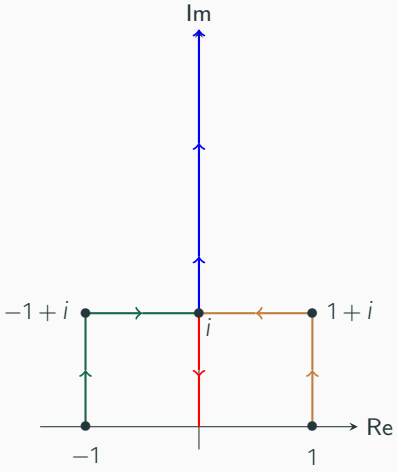

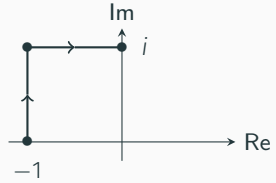

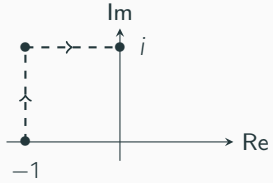

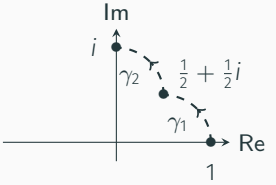

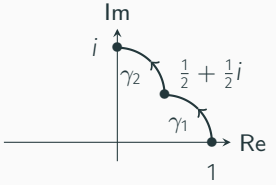

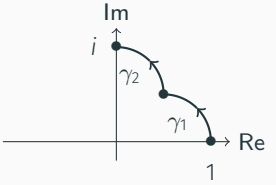

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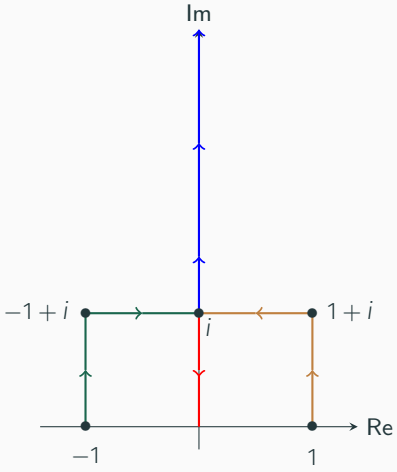

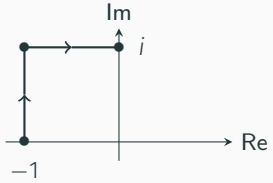

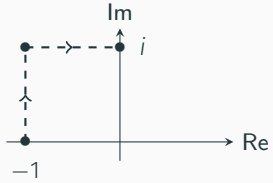

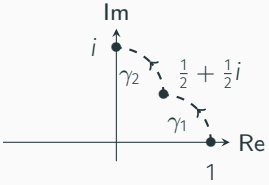

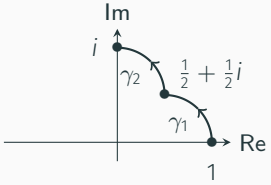

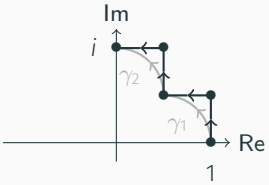

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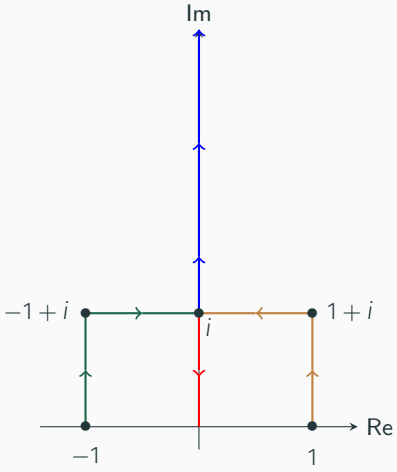

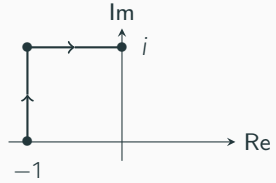

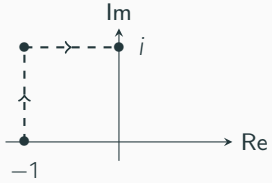

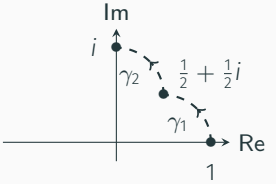

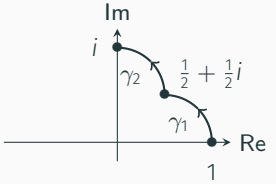

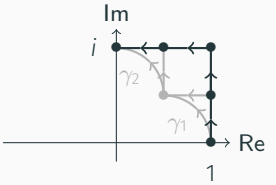

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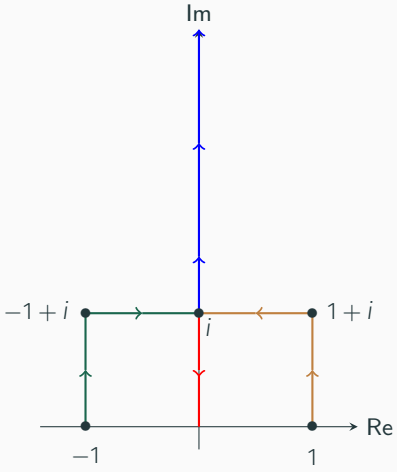

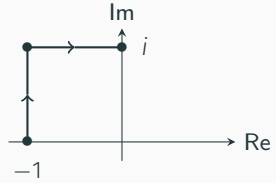

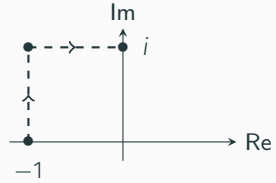

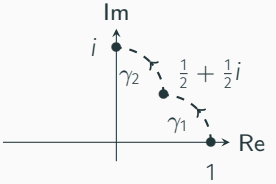

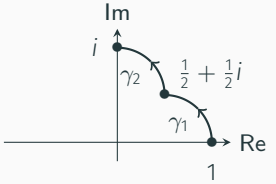

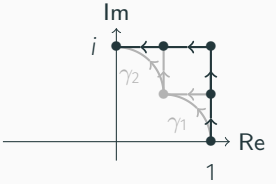

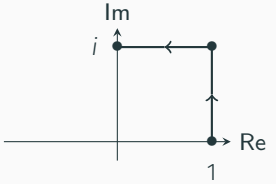

What did Gauss do?

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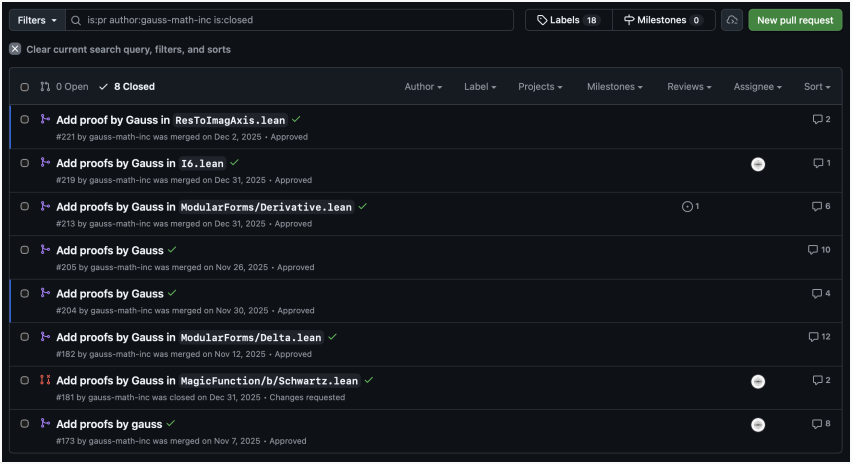

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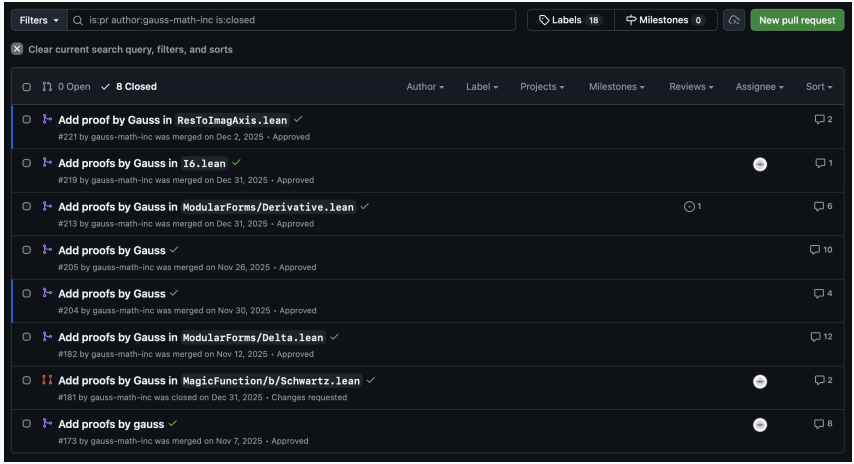

The Big Autoformalisation

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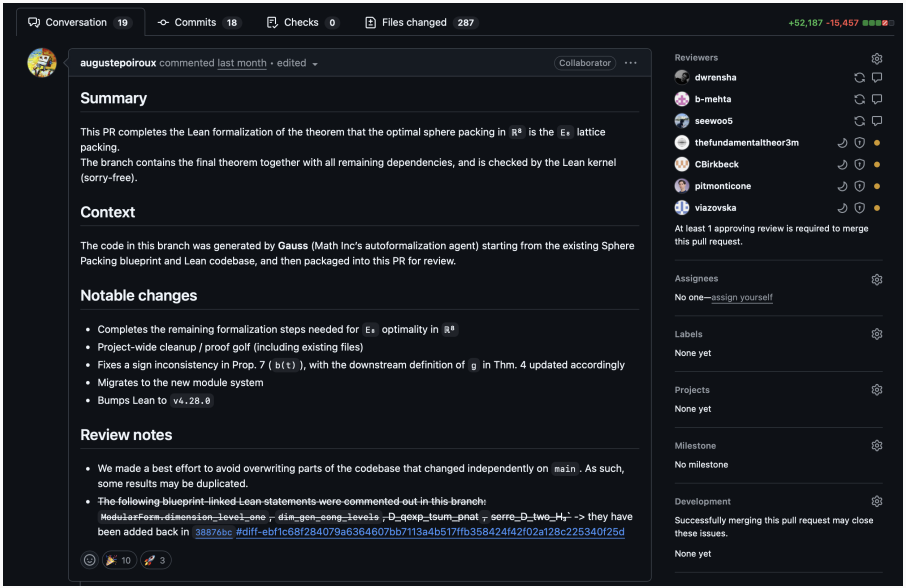

Conversation 19 · Commits 18 · Checks 0 · Files changed 287 · +52,187 -15,457

augustepoiroux commented last month • edited — Collaborator

Summary

This PR completes the Lean formalization of the theorem that the optimal sphere packing in $\mathbb{R}^8$ is the E_8 lattice packing.
The branch contains the final theorem together with all remaining dependencies, and is checked by the Lean kernel (sorry-free).

Context

The code in this branch was generated by **Gauss** (Math Inc's autoformalization agent) starting from the existing Sphere Packing blueprint and Lean codebase, and then packaged into this PR for review.

Notable changes

- Completes the remaining formalization steps needed for E_8 optimality in $\mathbb{R}^8$
- Project-wide cleanup / proof golf (including existing files)
- Fixes a sign inconsistency in Prop. 7 (`b(t)`), with the downstream definition of `g` in Thm. 4 updated accordingly
- Migrates to the new module system
- Bumps Lean to `v4.28.0`

Review notes

- We made a best effort to avoid overwriting parts of the codebase that changed independently on `main` . As such, some results may be duplicated.
- ~~The following blueprint-linked Lean statements were commented out in this branch:~~
  ~~`ModularForm.dimension_level_one`~~ , ~~`dim_gen_cong_levels`~~ , ~~`D_qexp_tsum_pnat`~~ , ~~`serre_D_two_H`~~ -> they have been added back in `3887ebc` #diff-ebf1c68f284079a6364607bb7113a4b517ffb358424f42f02a128c225340f25d

🎉 10 · 🚀 3

Reviewers
- dwrensha
- b-mehta
- seewoo5
- thefundamentaltheor3m
- CBirkbeck
- pitmonticone
- viazovska

At least 1 approving review is required to merge this pull request.

Assignees
No one—assign yourself

Labels
None yet

Projects
None yet

Milestone
No milestone

Development
Successfully merging this pull request may close these issues.
None yet

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- The file organisation is not the best, and definitions (especially unusual ones) often don't appear where they make the most sense.
- Some human-written code was also modified, and it is not yet completely clear that these modifications were good.
- **We are still reviewing, understanding, and refactoring the code. This will probably take a few months. In particular, none of the code has been merged yet.**

What *didn't* Gauss do?

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Finish the project!

Takeaways for Future Human-AI Collaboration

Why do we formalise mathematics?

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My personal motivation for formalising Viazovska's solution was not to verify it, but to understand it: **it was really about the journey, not the destination**. On recent events, Avigad writes [1, p.3]:

> *The formalization, on its own, is close to worthless, since* ***the correctness of Viazovska's result was never in doubt****. The participants embraced the project, rather, as a way of revisiting those results and* ***better understanding them****, and of* ***building libraries and infrastructure to support future work****. … [I]f AI can alleviate some of the tedium in formalization and help us enjoy and understand the mathematics, we will be better off, but* ***if it gets in the way of that enjoyment and understanding, we will have lost something important****.*

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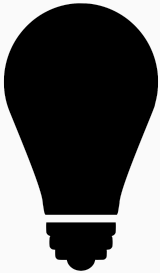

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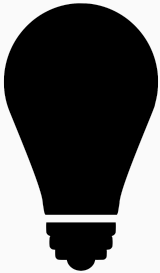

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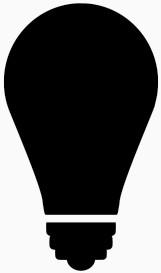

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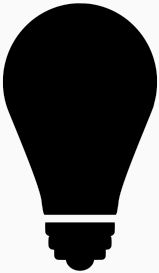

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WE NEED YOU!

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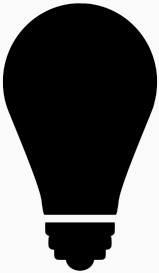

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Come to packathons:

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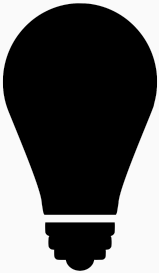

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WE NEED YOU!

Come to packathons: Tuesdays 10:30 Eastern Time at `cmu.zoom.us/my/sidharth.hariharan`.

Acknowledgements

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- Chris Birkbeck, Seewoo Lee, Gareth Ma, Bhavik Mehta, Maryna Viazovska

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Thank you very much! Merci beaucoup ! Merci vielmal! Grazie mille!

Questions?

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